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著者	Koshitani Shigeo
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THE PRINCIPAL 2-BLOCKS OF FINITE GROUPS WITH ABELIAN SYLOW 2-SUBGROUPS

By

Shigeo KOSHITANI

Introduction

Let G be a finite group, p a prime number and B a p -block of G with defect group D . There is an important problem in representation theory of finite groups that is to give a description of B when the structure of D is given. Concerning with this problem there are some successful results. E.C. Dade [9] proved his results when D is cyclic. R. Brauer [6] proved his results for the case where $p=2$ and D is dihedral by making use of his powerful methods ([3], [4], [5]). Using Brauer's methods J.B. Olsson [18] obtained his results when $p=2$ and D is generalized quaternion or quasidihedral. In [3, IV] R. Brauer investigated B when $p=2$ and D is elementary abelian of order 4.

In the present paper we study B when $p=2$ and B is the principal 2-block of G with an abelian Sylow 2-subgroup P . Let $e(G)=|N_G(P):C_G(P)|$. Let $B_0(G)$ be the principal 2-block of G , and let $O(G)$ and $O'(G)$ be the maximal normal subgroup of G of odd order and the minimal normal subgroup of G of odd index, respectively. By the results on finite groups with abelian Sylow 2-subgroups ([2], [16], [17], [20], [21]), the structure of $O'(G/O(G))$ is almost determined. In general, however, $B_0(G)$ is different from $B_0(S)$ where $S=O'(G/O(G))$. The main purpose of this paper is to investigate the relation between $B_0(G)$ and $B_0(S)$. In particular we shall prove that $B_0(G)$ is isomorphic to $B_0(S)$ for the cases where $e(G)=e(S)=\text{prime}$, 9 and 21.

In section 1 we shall state several lemmas and propositions which will be useful for our aim. One of them is Alperin's theorem on isomorphic principal blocks [1]. Let $S=O'(G/O(G))$. In section 2 we shall consider $B_0(G)$ for the case where $e(G)=2^m-1$. In particular, we shall prove that if G is nonsolvable and if $e(G)$ is prime then $e(G)=2^m-1$ for some $m\geq 2$ and $B_0(G)$ is isomorphic to $B_0(S)$. In sections 3 and 4 we shall investigate $B_0(G)$ for the cases when $e(G)=9$ and 21, respectively. Indeed, we shall prove that if $e(G)=e(S)=9$ or 21 then $B_0(G)$ is isomorphic to $B_0(S)$. It is noted that when $e(G)\neq e(S)$, $B_0(G)$ is not necessarily isomorphic to $B_0(S)$. In sections 5 and 6 we shall determine

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$B_0(G)$ when P is elementary abelian of order 8 and 16, respectively.

Throughout this paper we shall use the following notation. When S is a subset of G , $N_G(S)$ and $C_G(S)$ denote the normalizer and the centralizer of S in G , respectively. Specially, for each $x \in G$ we write $C_G(x)$ for $C_G(\{x\})$. If $x, y \in G$, we write x^y for $y^{-1}xy$. When S is a subset of G , $\langle S \rangle$ denotes the subgroup of G generated by S . When x_1, \dots, x_n are elements of G and S is a subset of G , we also write $\langle x_1, \dots, x_n, S \rangle$ for the subgroup of G generated by $\{x_1, \dots, x_n\} \cup S$. The cyclic group of order n is denoted Z_n for a positive integer n . We write G' and $Z(G)$ for the commutator subgroup of G and the center of G , respectively. We denote by $\text{Aut}(G)$ the group of all automorphisms of G . Let us denote by $O_{p'}(G)$ the maximal normal subgroup of G of order prime to p , and by $O^{p'}(G)$ the minimal normal subgroup of G of index prime to p . In particular, for $p=2$ we write $O(G)$ and $O'(G)$ for $O_2(G)$ and $O^{2'}(G)$, respectively. When P is an abelian Sylow 2-subgroup of G , we write $e(G)$ (or shortly e) for $|N_G(P) : C_G(P)|$. When B is a p -block of G , let us denote by $\text{Irr}(B)$ the set of all irreducible complex characters in B , by $\text{IBr}(B)$ the set of all irreducible Brauer characters in B , by $k(B)$ the number of elements of $\text{Irr}(B)$, by $k'(B)$ the number of elements of $\text{Irr}(B)$ with degree one, and by $l(B)$ the number of elements of $\text{IBr}(B)$. We write $B_0(G)$ (or shortly B_0) for the principal p -block of G , and for each $x \in G$ we write b_x for $B_0(C_G(x))$. When ϕ_1 and ϕ_2 are complex characters of G , let $(\phi_1, \phi_2) = (1/|G|) \sum_{g \in G} \phi_1(g) \phi_2(g^{-1})$, that is to say, (ϕ_1, ϕ_2) is the inner product of ϕ_1 and ϕ_2 . We write 1_G for the trivial complex (or Brauer) character of G . When H is a normal subgroup of G , $\phi|_H$ denotes the restriction of ϕ to H for a character ϕ of G , $W|_H$ denotes the restriction of W to H for a representation W of G , and $I_G(\tilde{\phi})$ denotes the inertial group of $\tilde{\phi}$ in G for a character $\tilde{\phi}$ of H , that is to say, $I_G(\tilde{\phi}) = \{g \in G \mid \tilde{\phi}^g = \tilde{\phi}\}$, where $\tilde{\phi}^g$ is the conjugate of $\tilde{\phi}$.

1. Preliminaries

In this section we state some lemmas and propositions which will be needed for our aim. We fix a prime number p and we consider p -modular representations of a finite group G .

LEMMA 1.1. *Let G be a finite group with a Sylow p -subgroup P , and let $K = O_{p'}(G)$, $\bar{G} = G/K$ and $\bar{P} = (PK)/K$. Then we have the following.*

- (i) $B_0(G) = B_0(\bar{G})$.
- (ii) $N_G(P)/C_G(P) \cong N_{\bar{G}}(\bar{P})/C_{\bar{G}}(\bar{P})$.

PROOF. We get (i) by [10, Theorem 65.2] and [11, V (4.3)]. Since $N_{\bar{G}}(\bar{P}) = (N_G(P) \cdot K)/K$ from [15, I 7.7 Hilfssatz (c)] and since $C_{\bar{G}}(\bar{P}) = (C_G(P) \cdot K)/K$ from [19, Lemma 2.2], we easily get (ii).

We shall frequently use the next four propositions in order to prove our main theorems.

PROPOSITION 1.2. (Brauer). *Let $G = QC_G(Q)$ where Q is a p -group, and let $\bar{G} = G/Q$. Then $l(B_0(G)) = l(B_0(\bar{G}))$.*

PROOF. See [10, Lemma 64.5 and Theorem 65.2(2)].

PROPOSITION 1.3 (Brauer). *Let H be a normal subgroup of G . If W is an ordinary or modular irreducible representation in $B_0(G)$, then any irreducible constituent of $W|_H$ lies in $B_0(H)$.*

PROOF. This is the special case of [3, I Lemma 1].

PROPOSITION 1.4 (Brauer). *Let H be a normal subgroup of G . Then for any $\tilde{\chi} \in \text{Irr}(B_0(H))$, there is some $\chi \in \text{Irr}(B_0(G))$ such that $(\chi|_H, \tilde{\chi}) \neq 0$.*

PROOF. This is the special case of [3, II Lemma 1].

PROPOSITION 1.5 (Brauer). *Let P be a Sylow p -subgroup of G , and let $P \cdot C_G(P) = P \times V$. Then $k'(B_0(G)) = |G : VG'|$.*

PROOF. See [3, IV Proposition (4G)].

Next, we state Alperin's theorems on isomorphic principal p -blocks which are very important for our aim.

Let F be an algebraically closed field of characteristic p and FG the group algebra of G over F . Let H be a normal subgroup of G with $p \nmid |G : H|$. We write $B_0(G) \cong B_0(H)$, if the category of all finitely generated FG -modules in $B_0(G)$ is isomorphic to the category of all finitely generated FH -modules in $B_0(H)$ and if the isomorphism is given by the restriction from G to H (cf. [1]).

PROPOSITION 1.6 (Alperin). *Let F be as above, and let P be a Sylow p -subgroup of G . If H is a normal subgroup of G which satisfies the conditions that $p \nmid |G : H|$, G/H is solvable and $G = H \cdot C_G(P)$, then we get the following.*

- (i) $B_0(G) \cong B_0(H)$.
- (ii) $A_0(G) \cong A_0(H)$ as F -algebras, where $A_0(G)$ and $A_0(H)$ are the block ideals

of FG and FH corresponding to $B_0(G)$ and $B_0(H)$, respectively.

PROOF. See [1, Theorems 1 and 2].

COROLLARY 1.7 (Alperin). *Let H be a normal subgroup of G of prime index q with $q \neq p$. Let $B_0 = B_0(G)$ and $b_0 = B_0(H)$. Assume that $k(B_0) = k(b_0)$ and $l(B_0) = l(b_0)$, and that $I_G(\tilde{\chi}) = G$ for every $\tilde{\chi} \in \text{Irr}(b_0)$. Then we have the following.*

- (i) *The correspondence $\text{Irr}(B_0) \rightarrow \text{Irr}(b_0)$ given by $\chi \mapsto \chi|_H$ is a bijection.*
- (ii) *The correspondence $\text{IBr}(B_0) \rightarrow \text{IBr}(b_0)$ given by $\phi \mapsto \phi|_H$ is a bijection.*
- (iii) $B_0 \cong b_0$.

PROOF. (i) Since $I_G(\tilde{\chi}) = G$ for every $\tilde{\chi} \in \text{Irr}(b_0)$, the correspondence is surjective by Clifford's theorem, [8, (53.17) Theorem] and Propositions 1.3 and 1.4. Since $k(B_0) = k(b_0)$, we obtain (i).

(ii) By (i), [1, Lemma 1] holds. Thus, by the proof of [1, Lemma 3], the correspondence is surjective. Hence (ii) holds since $l(B_0) = l(b_0)$.

(iii) Since [1, Lemmas 1 and 3] hold, we get (iii) by the proofs of Alperin's theorems [1, Theorems 1 and 2].

In the remainder of this paper we assume $p=2$ and let G and P be a finite group and its abelian Sylow 2-subgroup of order 2^n , respectively. We use the notation B_0 and e for $B_0(G)$ and $e(G)$, respectively.

COROLLARY 1.8 (Alperin). *Let H be a normal subgroup of G of odd prime index. Let $B_0 = B_0(G)$ and $b_0 = B_0(H)$. Assume that $k(B_0) = k(b_0)$ and $l(B_0) = l(b_0)$, and that H has an involution x such that $\chi(x) = \pm 1$ for every $\chi \in \text{Irr}(B_0)$ and $\tilde{\chi}(x) = \tilde{\chi}'(x) = \pm 1$ for all $\tilde{\chi}, \tilde{\chi}' \in \text{Irr}(b_0)$ with $\tilde{\chi}(1) = \tilde{\chi}'(1)$. Then $B_0 \cong b_0$.*

PROOF. By Clifford's theorem and Proposition 1.3, we have $\chi|_H \in \text{Irr}(b_0)$ for all $\chi \in \text{Irr}(B_0)$. Thus, by Proposition 1.4, $I_G(\tilde{\chi}) = G$ for all $\tilde{\chi} \in \text{Irr}(b_0)$. Thus the corollary is proved by Corollary 1.7 (iii).

LEMMA 1.9. *Let P be an abelian Sylow 2-subgroup of G . Suppose that $k(B_0) = |P|$ and that G has an involution x with $l(b_x) = 1$. Then $\chi(x) = \pm 1$ for all $\chi \in \text{Irr}(B_0)$.*

PROOF. Since $l(b_x) = 1$, b_x has the unique Cartan invariant $|P|$. Hence, by [10, Theorems 63.3(2), 63.2 and 65.4], we get $\sum \chi(x)^2 = |P|$ where the sum runs through all $\chi \in \text{Irr}(B_0)$. By [4, II (7A) and (4C)], $\chi(x)$ is a nonzero integer for every $\chi \in \text{Irr}(B_0)$ since $|x| = 2$. Therefore, the assumption $k(B_0) = |P|$ implies the

lemma.

PROPOSITION 1.10 (Bender, Janko, Janko-Thompson, Walter, Ward). *If G has abelian Sylow 2-subgroups, then $O'(G/O(G))$ is a direct product of an abelian 2-group and simple groups of one of the following types;*

- (1) *the special linear group $SL(2, 2^n)$ for $n \geq 2$,*
- (2) *the projective special linear group $L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$,*
- (3) *the Janko's first simple group J_1 ,*
- (4) *the simple group $R(q)$ of Ree type.*

PROOF. For groups of types (1) and (2), see [14, p. 40]. For J_1 see [16], and for $R(q)$ see [21]. The proposition is obtained from [2], [16], [17], [20] and [21].

In the rest of this paper we use the notation $SL(2, 2^n)$, $L_2(q)$, J_1 and $R(q)$ as in Proposition 1.10 (cf. [13, p. 415]). We also use the notation $GL(m, 2)$ for the general linear group (cf. [14, p. 40]).

The next lemma shows that Brauer's conjecture on heights of irreducible complex characters in p -blocks with abelian defect groups is affirmative for the principal 2-blocks of finite groups with abelian Sylow 2-subgroups.

LEMMA 1.11. *If G has abelian Sylow 2-subgroups, then all irreducible complex characters in $B_0(G)$ have height zero.*

PROOF. We may assume $O(G) = 1$ by Lemma 1.1. Let H be a normal subgroup of G of odd index. If $\chi \in \text{Irr}(B_0(G))$, then there is some $\tilde{\chi} \in \text{Irr}(B_0(H))$ with $\chi(1) = m\tilde{\chi}(1)$ for a positive integer m from Clifford's theorem and Proposition 1.3. By [8, (53.17) Theorem], m divides $|G : H|$. This shows that if $\tilde{\chi}(1)$ is odd then $\chi(1)$ is also odd. Thus, we may assume $O'(G) = G$. Then, by Proposition 1.10, we can write $G = Q \times (\prod S_i)$ where Q is an abelian 2-group and each S_i is a simple group of one of the following types;

- (i) $SL(2, 2^n)$ for $n \geq 2$,
- (ii) $L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$,
- (iii) J_1 ,
- (iv) $R(q)$.

When S_i is of type (i) or (ii), every $\chi \in \text{Irr}(B_0(S_i))$ has odd degree from [10, Theorems 38.2 and 38.1]. When S_i is of type (iii) or (iv), every $\chi \in \text{Irr}(B_0(S_i))$ has odd degree from [16, Lemma 5.1] and [21, Chap. I], respectively. These show that every $\chi \in \text{Irr}(B_0(G))$ has odd degree. This completes the proof.

The next three lemmas are useful in order to obtain $e=e(G)$.

LEMMA 1.12. *Let P be a Sylow 2-subgroup of G .*

(i) *If $G=SL(2, 2^n)$ for $n \geq 2$, then P is elementary abelian of order 2^n and $N_G(P)/C_G(P)$ is cyclic of order 2^n-1 .*

(ii) *If $G=L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$, then P is noncyclic of order 4 and $N_G(P)/C_G(P)$ is cyclic of order 3.*

(iii) *If $G=J_1$ or $R(q)$, then P is elementary abelian of order 8 and $N_G(P)/C_G(P)$ is noncyclic of order 21.*

PROOF. (i) By [14, Theorems 2.8.1 and 2.8.3], P is elementary abelian of order 2^n . Let $q=2^n$, and let F_q be the finite field of q elements. We may assume that $P=\left\{\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \mid f \in F_q\right\}$ (cf. the proof of [14, Theorem 2.8.3]). Clearly, $C_G(P)=P$. Let u be a generator of the multiplicative group $F_q-\{0\}$, and let $s=\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}$ in G . Then, $N_G(P)=\langle P, s \rangle$ and s has order $q-1$. Hence we get that $N_G(P)/P$ is cyclic of order $q-1$.

(ii) P is noncyclic of order 4 from [14, Lemma 15.1.1]. Hence $\text{Aut}(P)$ is isomorphic to the symmetric group of degree 3. Since G is not 2-nilpotent, we get (ii).

(iii) If $G=J_1$, we obtain (iii) from [16, VI p. 160]. Assume $G=R(q)$. By [21, p. 63], P is elementary abelian of order 8 and $|N_G(P):C_G(P)|=21$. Then we know that $N_G(P)/C_G(P)$ is noncyclic since $\text{Aut}(P) \cong GL(3, 2) \hookrightarrow GL(4, 2) \cong A_8$ from [15, II 2.5 Satz] where A_8 is the alternating group of degree 8.

LEMMA 1.13. (i) *$GL(4, 2) \cong A_8$, the alternating group of degree 8.*

(ii) *If H is a subgroup of A_8 of odd order, then $|H|=1, 3, 5, 7, 9, 15$ or 21 .*

(iii) *A_8 has subgroups of orders 1, 3, 5, 7, 9, 15 and 21, and the subgroups of order 9 and the subgroups of order 21 are noncyclic.*

PROOF. (i) We have already showed (i) in the proof of Lemma 1.12(iii).

(ii) Since $|A_8|=2^6 \cdot 3^2 \cdot 5 \cdot 7$, $|H|=1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105$ or 315 . Since the groups of order 35 are cyclic, $|H| \neq 35$. By elementary calculations, A_8 has no subgroups of order 45, so that $|H| \neq 45$. Similarly, $|H| \neq 63$. If $|H|=105$, then H has an element of order 35. Evidently, this is a contradiction. Hence $|H| \neq 105$. If $|H|=315$, then H has an element of order 35, and this is a contradiction. So that $|H| \neq 315$.

(iii) By Sylow's theorem, A_8 has subgroups of orders 3, 5, 7 and 9. Since A_8 has no elements of order 9, Sylow 3-subgroups of A_8 are noncyclic of order

9. If $G = SL(2, 2^4)$, then P is elementary abelian of order 16 and $N_G(P)/C_G(P)$ is cyclic of order 15 from Lemma 1.12(i). Thus, by (i), A_8 has subgroups of order 15. Let $H = \langle (124)(536), (1234567) \rangle$. Then H is a noncyclic subgroup of A_8 of order 21. Since A_8 has no elements of order 21, all subgroups of A_8 of order 21 are noncyclic.

LEMMA 1.14. (i) *If H is a subgroup of $GL(3, 2)$ of odd order, then $|H| = 1, 3, 7$ or 21 .*

(ii) *$GL(3, 2)$ has subgroups of orders 1, 3, 7 and 21, and the subgroups of order 21 are noncyclic.*

PROOF. (i) By [10, Lemma 35.2(1)], $|GL(3, 2)| = 2^3 \cdot 3 \cdot 7$. So that we easily get (i).

(ii) By the proof of (i) and Sylow's theorem, $GL(3, 2)$ has subgroups of orders 3 and 7. By Lemma 1.12(iii), $GL(3, 2)$ has noncyclic subgroups of order 21. Since $GL(3, 2) \hookrightarrow GL(4, 2)$, all subgroups of $GL(3, 2)$ of order 21 are noncyclic from Lemma 1.13(i) and (iii).

The next two lemmas are useful in order to determine B_0 when Sylow 2-subgroups of G are elementary abelian of order 8 or 16.

LEMMA 1.15. *Let P be an abelian Sylow 2-subgroup of G , and let $B_0 = B_0(G)$. Assume that G has an involution x with $l(b_x) = 1$.*

(1) *If $|P| = 8$, then $k(B_0) = 8$.*

(2) *If $|P| = 16$, then $k(B_0) = 8$ or 16 .*

PROOF. Let $\{\chi_1, \dots, \chi_{k(B_0)}\} = \text{Irr}(B_0)$. Since $l(b_x) = 1$, by [10, Theorems 63.2 and 65.4], for each χ_i let d_{ii}^x be the generalized decomposition number of B_0 relative to x . By Lemma 1.11 and [4, II (7A) and (4C)], every d_{ii}^x is an odd integer. Since b_x has the unique Cartan invariant $|P|$, by [10, Theorem 63.3], $\sum_{i=1}^{k(B_0)} (d_{ii}^x)^2 = |P|$. These imply (1) and (2).

LEMMA 1.16. *Let $G = L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$, and let $B_0 = B_0(G)$. Then we have the following.*

(i) *$l(B_0) = 3$ and the degrees of all irreducible Brauer characters in B_0 are 1, $(q-1)/2$ and $(q-1)/2$.*

(ii) *The decomposition matrix of B_0 is as follows:*

$$\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{array}
\qquad
\begin{array}{ccc}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 1 & 1.
\end{array}$$

$$3 < q \equiv 3 \pmod{8} \qquad 3 < q \equiv 5 \pmod{8}$$

PROOF. Since G is not 2-nilpotent, $l(B_0) > 1$ from [10, Corollary 65.3]. Thus $k(B_0) = 4$ and $l(B_0) = 3$ by [3, IV Proposition (7D)].

Case 1. $3 < q \equiv 3 \pmod{8}$: Let $\text{Irr}(B_0) = \{\chi_1, \dots, \chi_4\}$. By [10, Theorem 38.1], we may assume $\chi_1 = 1_G$, $\chi_2(1) = \chi_3(1) = (q-1)/2$ and $\chi_4(1) = q$. By [14, Theorem 2.8.2], G has a Frobenius subgroup E of order $q(q-1)/2$. We know the character tables of E and $L_2(q)$ from [10, Theorems 13.8 and 38.1]. Thus, by [8, §84 Exercise 2], $\chi_2|_{G_0}$ and $\chi_3|_{G_0}$ are both irreducible Brauer characters of G , where $\chi_i|_{G_0}$ is the restriction of χ_i to the set G_0 of all $2'$ -elements of G . Since $\chi_2 \neq \chi_3$ on G_0 , and since $\chi_4 = \chi_1 + \chi_2 + \chi_3$ on G_0 , we know (i) and the decomposition matrix of B_0 . Case 2. $3 < q \equiv 5 \pmod{8}$: As in Case 1 we can prove the lemma.

REMARK 1. If G has an abelian Sylow 2-subgroup P and if $e(G) = 1$, then $B_0(G) \cong B_0(P)$ since G is 2-nilpotent by [10, Theorem 18.7].

2. The case $e = 2^m - 1$

In this section we consider the case when $e = 2^m - 1$ for $m \geq 2$. We use the notation G, P, n, e and B_0 as before, that is to say, P is an abelian Sylow 2-subgroup of G with order 2^n ($n \geq 2$), $e = e(G)$ and $B_0 = B_0(G)$. To begin with we state the next three lemmas which will be needed for the main result of this section.

LEMMA 2.1. *Let S be a normal subgroup of G of odd index such that $S \cong SL(2, 2^n)$ for some $n \geq 3$. Assume $e = 2^n - 1$. Then $B_0 \cong B_0(S)$.*

PROOF. We may assume $S = SL(2, 2^n)$. There are an element $t \in N_S(P)$ and an involution $x \in P$ such that $N_S(P) = \langle t, C_S(P) \rangle$ and $P = \{1, x, x^t, \dots, x^{t^{2^n-2}}\}$ (cf. the proof of Lemma 1.12(i)). Since $e = 2^n - 1$, $N_G(P) = \langle t, C_G(P) \rangle$. Clearly $y^t \neq y$ for all $y \in P - \{1\}$, so that $N_M(P) = C_M(P)$ where $M = C_G(x)$. Hence M is 2-nilpotent from [10, Theorem 18.7]. Thus, by [10, Corollary 65.3], $l(b_x) = l(B_0(M)) = 1$. Now, we prove the lemma by induction on $|G|$. Suppose $G \neq S$. Since $|G/S|$ is odd, by [12, Theorem], G has a normal subgroup H of odd prime index l with $S \subseteq H$. Let $b_0 = B_0(H)$. By induction, $b_0 \cong B_0(S)$. Hence, by the character table of $SL(2, 2^n)$ [10, Theorem 38.2], we get

	1	x	
1_H	1	1	
$\tilde{\theta}_i$	$2^n - 1$	-1	for $i=1, \dots, 2^{n-1}$
$\tilde{\chi}_j$	$2^n + 1$	1	for $j=1, \dots, 2^{n-1} - 1$

where $\{1_H, \tilde{\theta}_i, \tilde{\chi}_j | i=1, \dots, 2^{n-1}; j=1, \dots, 2^{n-1}-1\} = \text{Irr}(b_0)$. Let $C_G(P) = P \times V$. If $G = VH$, then $G = C_G(P) \cdot H$, so that $B_0 \cong b_0$ from Proposition 1.6. Hence we may assume $G \neq VH$. Then $H = VH$, so that $C_H(P) = P \times V$. Thus, by Proposition 1.5, $k'(b_0) = |H : VH'|$. Since $b_0 \cong B_0(S)$, $k'(b_0) = 1$. Thus, $H = VH'$. This implies $H = VG'$ since G/H is cyclic. Hence $k'(B_0) = 1$ from Proposition 1.5. By Clifford's theorem and Proposition 1.3, for each $\chi \in \text{Irr}(B_0)$ one of the following five cases occurs:

- (a) $\chi|_H = 1_H$,
- (b) $\chi|_H = \tilde{\theta}_i$ for some i ,
- (c) $\chi|_H = \tilde{\theta}_{i_1} + \dots + \tilde{\theta}_{i_l}$ for $i_1 < \dots < i_l$, and all $\tilde{\theta}_{i_k}$ are G -conjugate,
- (d) $\chi|_H = \tilde{\chi}_j$ for some j ,
- (e) $\chi|_H = \tilde{\chi}_{j_1} + \dots + \tilde{\chi}_{j_l}$ for $j_1 < \dots < j_l$, and all $\tilde{\chi}_{j_k}$ are G -conjugate.

Since $k'(b_0) = 1$, for each $\chi \in \text{Irr}(B_0)$ $\chi(1) = 1$ if and only if $\chi|_H = 1_H$. Let r, s, u and v be the numbers of $\chi \in \text{Irr}(B_0)$ of types (b), (c), (d) and (e), respectively. Since $l(b_x) = 1$, as in the proof of Lemma 1.9, $\sum \chi(x)^2 = 2^n$ where the sum runs through all $\chi \in \text{Irr}(B_0)$. This shows $l + r + sl^2 + u + vl^2 = 2^n$. On the other hand, by Proposition 1.4, for every $\tilde{\chi} \in \text{Irr}(b_0)$ there is some $\chi \in \text{Irr}(B_0)$ with $(\chi|_H, \tilde{\chi}) \neq 0$. So that $k(b_0) \leq 1 + r + sl + u + vl$. Since $k(b_0) = 2^n$, we have a contradiction. This completes the proof.

REMARK 1. We can not remove the assumption $e = 2^n - 1$ in Lemma 2.1. Indeed, let $S = SL(2, 8)$ and $P = \left\{ \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \mid f \in F_8 \right\}$ where F_8 is the finite field of 8 elements. Let u be a generator of the multiplicative group $F_8 - \{0\}$. There is an automorphism h of F_8 with $h(u) = u^2$. For each $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S$ let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^h = \begin{pmatrix} h(a) & h(b) \\ h(c) & h(d) \end{pmatrix}$. Then we can consider $h \in \text{Aut}(S)$ and $h|_P \in \text{Aut}(P)$ where $h|_P$ is the restriction of h to P . Hence there is a semi-direct product G of its normal subgroup S by $\langle h \rangle$. Then $O'(G) = S = SL(2, 8)$ and $e(G) = 21 \neq 2^3 - 1$. By [10, Theorem 38.2], $l(B_0(S)) = 7$. But we shall afterwards show that $l(B_0(G)) = 5$, and this shows $B_0(G) \not\cong B_0(S)$.

LEMMA 2.2. Let S be a normal subgroup of G of odd index such that $S \cong L_2(q) \times (P/(Z_2 \times Z_2))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$, or $S \cong SL(2, 2^m)$

$\times (P/(\underbrace{Z_2 \times \cdots \times Z_2}_m))$ for some $m \geq 3$. Assume $e=e(S)$. Then $k(B_0)=2^n$ and $l(B_0)=e$.

PROOF. Let $L=L_2(q)$ for $m=2$, and let $L=SL(2, 2^m)$ for $m \geq 3$. Let R be a Sylow 2-subgroup of L . We can write $S=L \times Q$ and $P=R \times Q$. We use induction on n . If $n=m=2$, then the lemma is proved by [3, IV Proposition (7D)]. If $n=m \geq 3$, by Lemma 2.1, $B_0 \cong B_0(S)$, so that $k(B_0)=2^n$ and $l(B_0)=2^n-1=2^m-1$ (cf. [10, Theorem 38.2]). Next, suppose $n > m$. There are an element $t \in N_L(R)$ and an involution $x \in R$ such that $N_L(R)=\langle t, C_L(R) \rangle$ and $R=\{1, x, x^t, \dots, x^{t^{2^m-2}}\}$. Since $e=e(S)$, $N_G(P)=\langle t, C_G(P) \rangle$. Let $Q=\{1=y_1, y_2, \dots, y_{2^n-m}\}$. Then, by [10, Lemma 18.5], the G -conjugate classes of P are as follows:

$$\begin{aligned} & \{1\} \\ & \{y_i\} \text{ for } i=2, \dots, 2^{n-m} \\ & \{xy_i, x^t y_i, \dots, x^{t^{2^m-2}} y_i\} \text{ for } i=1, \dots, 2^{n-m}. \end{aligned}$$

Then, by [10, Theorems 68.4 and 65.4],

$$k(B_0)=l(B_0)+\sum_{i=2}^{2^{n-m}} l(b_{y_i})+\sum_{i=1}^{2^{n-m}} l(b_{xy_i}).$$

Fix any i with $2 \leq i \leq 2^{n-m}$, and let $M=C_G(y_i)$. Since $y_i \in Z(S)$, let $\bar{S}=S/\langle y_i \rangle$. Similarly, let $\bar{M}=M/\langle y_i \rangle$, $\bar{P}=P/\langle y_i \rangle$ and $\bar{Q}=Q/\langle y_i \rangle$. Since $\bar{S} \cong L \times \bar{Q}$, we get $e(\bar{S})=e(L)=2^m-1$. Since $\bar{S} \subseteq \bar{M}$, the canonical homomorphism $N_{\bar{S}}(\bar{P})/C_{\bar{S}}(\bar{P}) \rightarrow N_{\bar{M}}(\bar{P})/C_{\bar{M}}(\bar{P})$ is monomorphic. This shows $(2^m-1) | e(\bar{M})$. On the other hand, by [15, I 7.7 Hilfssatz (c)], we get $N_{\bar{M}}(\bar{P})=(N_M(P) \cdot \langle y_i \rangle)/\langle y_i \rangle$. This implies that the canonical homomorphism $N_M(P)/C_M(P) \rightarrow N_{\bar{M}}(\bar{P})/C_{\bar{M}}(\bar{P})$ is epimorphic. Hence $e(\bar{M}) | e(M)$. Since $S \subseteq M \subseteq G$ and $e=e(S)=2^m-1$, we have $e(M)=e(S)=2^m-1$ by considering the canonical monomorphisms as above. Thus $e(\bar{M})=2^m-1$. Hence we get $l(B_0(\bar{M}))=2^m-1$ by induction. Thus $l(b_{y_i})=l(B_0(M))=2^m-1$ from Proposition 1.2. We may assume $O(G)=1$ by Lemma 1.1. Since $Q \neq 1$, there is an involution $y_j \in Q$. By Z^* -theorem [10, Theorem 67.1], $y_j \in Z(G)$. Hence $l(B_0)=l(b_{y_j})=2^m-1$. Next, we consider $l(b_{xy_i})$ for each $i=1, \dots, 2^{n-m}$. For an integer k it is seen that $(xy_i)^{t^k}=xy_i$ if and only if $(2^m-1) | k$. Hence $N_U(P)=C_U(P)$ where $U=C_G(xy_i)$. Then U is 2-nilpotent from [10, Theorem 18.7], so that $l(b_{xy_i})=l(B_0(U))=1$ by [10, Corollary 65.3]. These imply $k(B_0)=2^n$.

LEMMA 2.3. Assume as in Lemma 2.2. Then $B_0 \cong B_0(S)$.

PROOF. We use the same notation as in the proof of Lemma 2.2. We prove the lemma by induction on $|G|$. Suppose $G \neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index with $S \subseteq H$. Let $b_0=B_0(H)$. By induction, $b_0 \cong B_0(S)$. It follows from Lemma 2.2 that $k(B_0)=k(b_0)=2^n$ and that

$l(B_0)=l(b_0)=2^m-1$. By the proof of Lemma 2.2, there is an involution $x \in G$ with $l(b_x)=1$. Hence $\chi(x)=\pm 1$ for all $\chi \in \text{Irr}(B_0)$ from Lemma 1.9. Thus, by Corollary 1.8, it is sufficient to show that

$$(*) \quad \begin{aligned} & \text{if } \tilde{\chi}, \tilde{\chi}' \in \text{Irr}(b_0) \text{ with } \tilde{\chi}(1)=\tilde{\chi}'(1), \\ & \text{then } \tilde{\chi}(x)=\tilde{\chi}'(x)=\pm 1. \end{aligned}$$

Let $\{\theta_1, \dots, \theta_{2^n-m}\}$ be the set of all irreducible complex characters of Q .

Case 1. $m=2$: By the character table of $L_2(q)$ (cf. [10, Theorem 38.1]), we can write

$$\begin{array}{ccc} & 1 & x \\ \zeta_1 & 1 & 1 \\ \zeta_2 & (q+\varepsilon)/2 & -\varepsilon \\ \zeta_3 & (q+\varepsilon)/2 & -\varepsilon \\ \zeta_4 & q & \varepsilon, \end{array} \quad \varepsilon = \begin{cases} -1 & \text{if } q \equiv 3 \pmod{8} \\ 1 & \text{if } q \equiv 5 \pmod{8} \end{cases}$$

where $\{\zeta_1, \dots, \zeta_4\} = \text{Irr}(B_0(L_2(q)))$. Since $b_0 \cong B_0(S)$ and since $S = L_2(q) \times Q$, we may write $\text{Irr}(b_0) = \{\tilde{\chi}_{ij} \mid i=1, \dots, 4; j=1, \dots, 2^{n-2}\}$ such that $\tilde{\chi}_{ij}|_S = \zeta_i \theta_j$ for all i, j . Then

$$\tilde{\chi}_{ij}(1) = \begin{cases} 1 & \text{for } i=1 \\ (q+\varepsilon)/2 & \text{for } i=2, 3 \\ q & \text{for } i=4 \end{cases}$$

and

$$\tilde{\chi}_{ij}(x) = \begin{cases} 1 & \text{for } i=1 \\ -\varepsilon & \text{for } i=2, 3 \\ \varepsilon & \text{for } i=4. \end{cases}$$

These imply (*).

Case 2. $m \geq 3$: By the character table of $SL(2, 2^m)$ (cf. [10, Theorem 38.2]), we know

$$\begin{array}{ccc} & 1 & x \\ 1 & 1 & 1 \\ \tilde{\theta}_i & 2^m-1 & -1 \quad \text{for } i=1, \dots, 2^{m-1} \\ \tilde{\chi}_j & 2^m+1 & 1 \quad \text{for } j=1, \dots, 2^{m-1}-1 \end{array}$$

where $\{1, \tilde{\theta}_i, \tilde{\chi}_j \mid i=1, \dots, 2^{m-1}; j=1, \dots, 2^{m-1}-1\} = \text{Irr}(B_0(SL(2, 2^m)))$. Using this we can show (*) as in Case 1. This completes the proof.

Now, the above lemmas imply the next main result of this section.

THEOREM 2.4. *Let P be an abelian Sylow 2-subgroup of G . Assume that e is prime. Then we have the following.*

- (1) $k(B_0)=e$. And if G is nonsolvable, then $k(B_0)=|P|$.
- (2) When G is nonsolvable, one of the following holds:
 - (i) $e=3$, and $B_0 \cong B_0(L_2(q) \times (P/(Z_2 \times Z_2)))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$,
 - (ii) $e=2^m-1$ for some $m \geq 3$, and $B_0 \cong B_0(SL(2, 2^m) \times (P/(\underbrace{Z_2 \times \cdots \times Z_2}_m)))$.

PROOF. We can assume $O(G)=1$ by Lemma 1.1. Let $S=O'(G)$. Firstly assume that S is solvable. Then $S=P$, so that $C_G(P)=P$. Hence G is a semi-direct product of its normal subgroup P by Z_e . This shows $k(B_0)=e$. So it is enough to consider the case where G is nonsolvable. Since e is prime, $e=e(S)$. By Proposition 1.10 and Lemma 1.12, one of the following two cases occurs:

- (i) $e(S)=3$, and $S \cong L_2(q) \times (P/(Z_2 \times Z_2))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$,
- (ii) $e(S)=2^m-1$ for some $m \geq 3$, and $S \cong SL(2, 2^m) \times (P/(\underbrace{Z_2 \times \cdots \times Z_2}_m))$.

Hence we obtain (1) and (2) from Lemmas 2.2 and 2.3, respectively.

REMARK 2. For the case where G is solvable, the latter half of Theorem 2.4(1) does not hold in general. Indeed, let P be an elementary abelian group of order 16 with $P=\langle x, y, z, w \rangle$. Let $t \in \text{Aut}(P)$ such that $x^t=y$, $y^t=xy$, $z^t=w$ and $w^t=zw$. There is a semi-direct product G of its normal subgroup P by $\langle t \rangle$. Then G is solvable and $e=|G:P|=3$. Since $u^t \neq u$ for all $u \in P - \{1\}$, we shall show that $k(B_0)=8 \neq 16$ (cf. Proposition 6.1). As another example, let P be the same as above, and let $t \in \text{Aut}(P)$ with $|t|=5$. If G is a semi-direct product of P by $\langle t \rangle$ and G is not the direct product $P \times Z_5$, then we shall show that $k(B_0)=8 \neq 16$ (cf. Proposition 6.3).

3. The case $e=9$

In this section we consider the case when $e=e(S)=9$, where $S=O'(G/O(G))$. We use the notation G, P, n, e and B_0 as in §2.

LEMMA 3.1. *Let P be an elementary abelian Sylow 2-subgroup of G of order 16. If $e=9$, then $k(B_0)=16$ and $l(B_0)=9$.*

PROOF. By Lemma 1.13, $\text{Aut}(P)$ has noncyclic Sylow 3-subgroups of order 9. Hence we may assume that $N_G(P)=\langle s, t, C_G(P) \rangle$ for some $s, t \in N_G(P)$, $P=\langle x, y, z, w \rangle$, $x^s=x$, $y^s=y$, $z^s=w$, $w^s=zw$, $x^t=y$, $y^t=xy$, $z^t=z$ and $w^t=w$. By [10, Lemma 18.5 and Theorems 68.4 and 65.4],

$$k(B_0) = l(B_0) + l(b_x) + l(b_z) + l(b_{xz}).$$

Since $e(C_G(xz))=1$, $l(b_{xz})=1$ from [10, Theorem 18.7 and Corollary 65.3]. Since $e(C_G(x))=e(C_G(z))=3$, it follows from Theorem 2.4 that $l(b_x)=l(b_z)=3$. By [10, Corollary 65.3], $l(B_0) \geq 2$ since $e=9$. Hence, by Lemma 1.15(2), $k(B_0)=16$, so that $l(B_0)=9$.

LEMMA 3.2. *Let S be a normal subgroup of G of odd index such that $S \cong L_2(q) \times L_2(q') \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q, q' > 3$ with $q \equiv 3$ or $5 \pmod{8}$ and $q' \equiv 3$ or $5 \pmod{8}$. If $e=9$, then $k(B_0)=2^n$ and $l(B_0)=9$.*

PROOF. We may assume $S = L_2(q) \times L_2(q') \times Q$ where $Q \cong P/(Z_2 \times Z_2 \times Z_2 \times Z_2)$. We use induction on n . If $n=4$, Sylow 2-subgroups of G are elementary abelian of order 16, so that the lemma is proved by Lemma 3.1. Suppose $n > 4$. Let R_1 and R_2 be Sylow 2-subgroups of $L_2(q)$ and $L_2(q')$, respectively. We may assume $P = R_1 \times R_2 \times Q$. We can write $R_1 = \{1, x, x^s, x^{s^2}\}$ for some $s \in L_2(q)$ and for an involution $x \in R_1$. Similarly, $R_2 = \{1, y, y^t, y^{t^2}\}$ for some $t \in L_2(q')$ and for an involution $y \in R_2$. Since $e=e(S)=9$, we know that $N_G(P) = \langle s, t, C_G(P) \rangle$ and that $N_G(P)/C_G(P)$ is elementary abelian of order 9. Let $Q = \{1 = z_1, z_2, \dots, z_{2^{n-4}}\}$. By [10, Lemma 18.5], $\{z_i, xz_i, yz_i, xyz_i \mid i=1, \dots, 2^{n-4}\}$ is the set of all representatives of G -conjugate classes of P . Thus, by [10, Theorems 68.4 and 65.4],

$$k(B_0) = l(B_0) + \sum_{i=2}^{2^{n-4}} l(b_{z_i}) \\ + \sum_{i=1}^{2^{n-4}} \{l(b_{xz_i}) + l(b_{yz_i}) + l(b_{xyz_i})\}.$$

As in the proof of Lemma 2.2, by induction, we get $l(b_{z_i})=9$ for all $i=2, \dots, 2^{n-4}$. By Lemma 1.1, we may assume $O(G)=1$. Since $Q \neq 1$, as in the proof of Lemma 2.2, by making use of Z^* -theorem [10, Theorem 67.1], we have $l(B_0)=9$. Since $s \in C_G(xz_i)$ and since $t \in C_G(yz_i)$, we obtain $e(C_G(xz_i))=3$. Hence $l(b_{xz_i})=3$ for all $i=1, \dots, 2^{n-4}$ from Theorem 2.4(1). Similarly, by Theorem 2.4(1), $l(b_{yz_i})=3$ for all $i=1, \dots, 2^{n-4}$. Fix any i with $1 \leq i \leq 2^{n-4}$. For integers j and k , it is seen that $(xyz_i)^{s^j t^k} = xyz_i$ if and only if $3 \mid j$ and $3 \mid k$. Hence as in the proof of Lemma 2.2, $l(b_{xyz_i})=1$ for all $i=1, \dots, 2^{n-4}$. Thus $k(B_0)=2^n$. This finishes the proof.

LEMMA 3.3. *Assume as in Lemma 3.2. Then $B_0 \cong B_0(S)$.*

PROOF. We use the same notation as in the proof of Lemma 3.2. We prove the lemma by induction on $|G|$. Assume $G \neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index with $S \subseteq H$. Let $b_0 = B_0(H)$. By

induction, $b_0 \cong B_0(S)$. By the proof of Lemma 3.2, there is an involution $xy \in G$ with $l(b_{xy})=1$. It follows from Lemmas 3.2 and 1.9 that $\chi(xy)=\pm 1$ for all $\chi \in \text{Irr}(B_0)$. By Lemma 3.2, $k(B_0)=k(b_0)$ and $\ell(B_0)=\ell(b_0)$. Thus, by Corollary 1.8, it is enough to prove that

$$(*) \quad \begin{array}{l} \text{if } \tilde{\chi}, \tilde{\chi}' \in \text{Irr}(b_0) \text{ with } \tilde{\chi}(1)=\tilde{\chi}'(1), \\ \text{then } \tilde{\chi}(xy)=\tilde{\chi}'(xy)=\pm 1. \end{array}$$

As in the proof of Lemma 2.3 we know the character tables of $L_2(q)$ and $L_2(q')$. Thus we can write

$$\begin{array}{ccc} & 1 & x \\ \eta_1 & 1 & 1 \\ \eta_2 & (q+\varepsilon)/2 & -\varepsilon \\ \eta_3 & (q+\varepsilon)/2 & -\varepsilon \\ \eta_4 & q & \varepsilon \end{array} \quad \varepsilon = \begin{cases} -1 & \text{if } q \equiv 3 \pmod{8} \\ 1 & \text{if } q \equiv 5 \pmod{8} \end{cases}$$

where $\{\eta_1, \eta_2, \eta_3, \eta_4\} = \text{Irr}(B_0(L_2(q)))$, and

$$\begin{array}{ccc} & 1 & y \\ \zeta_1 & 1 & 1 \\ \zeta_2 & (q'+\varepsilon')/2 & -\varepsilon' \\ \zeta_3 & (q'+\varepsilon')/2 & -\varepsilon' \\ \zeta_4 & q' & \varepsilon' \end{array} \quad \varepsilon' = \begin{cases} -1 & \text{if } q' \equiv 3 \pmod{8} \\ 1 & \text{if } q' \equiv 5 \pmod{8} \end{cases}$$

where $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} = \text{Irr}(B_0(L_2(q')))$. Let $\{\theta_1, \dots, \theta_{2^n-4}\}$ be the set of all irreducible complex characters of Q . Since $b_0 \cong B_0(S)$, we may write $\text{Irr}(b_0) = \{\tilde{\chi}_{ijk} \mid i=1, \dots, 4; j=1, \dots, 4; k=1, \dots, 2^{n-4}\}$ such that $\tilde{\chi}_{ijk}|_S = \eta_i \zeta_j \theta_k$ for all i, j, k .

Case 1. $\varepsilon=-1$ and $\varepsilon'=1$: In order to show (*) it is enough to prove that $\{1, (q-1)/2, q', (q-1)q'/2, q(q'+1)/2\} \cap \{(q'+1)/2, q, (q-1)(q'+1)/4, qq'\} = \emptyset$ since $\tilde{\chi}_{ijk}(1) = \eta_i(1)\zeta_j(1)$ and $\tilde{\chi}_{ijk}(xy) = \eta_i(x)\zeta_j(y)$ for all i, j, k . We can prove it.

Case 2. $\varepsilon=\varepsilon'=-1$: We know that $\{1, (q-1)/2, (q'-1)/2, (q-1)(q'-1)/4, qq'\} \cap \{q, q', (q-1)q'/2, q(q'-1)/2\} = \emptyset$. This implies (*) as in Case 1.

Case 3. $\varepsilon=\varepsilon'=1$: Since $\{1, q, q', (q+1)(q'+1)/4, qq'\} \cap \{(q+1)/2, (q'+1)/2, (q+1)q'/2, q(q'+1)/2\} = \emptyset$, we can show (*). This completes the proof of the lemma.

The above lemmas imply the next main result of this section.

THEOREM 3.4. *Let P be an abelian Sylow 2-subgroup of G . Assume $e=e(S)$*

$=9$, where $S=O'(G/O(G))$. Then we have the following.

- (1) $k(B_0)=|P|$ and $l(B_0)=9$.
- (2) $B_0 \cong B_0(L_2(q) \times L_2(q') \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2)))$ for some $q, q' > 3$ with $q \equiv 3$ or $5 \pmod{8}$ and $q' \equiv 3$ or $5 \pmod{8}$.

PROOF. We may assume $O(G)=1$ by Lemma 1.1. Since $e(S)=9$, by Proposition 1.10 and Lemma 1.12, we get that $S \cong L_2(q) \times L_2(q') \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q, q' > 3$ with $q \equiv 3$ or $5 \pmod{8}$ and $q' \equiv 3$ or $5 \pmod{8}$. Hence we obtain (1) and (2) from Lemmas 3.2 and 3.3, respectively.

4. The case $e=21$

In this section we deal with the case when $e=e(S)=21$, where $S=O'(G/O(G))$. As in §1, let J_1 and $R(q)$ be the Janko's first simple group and the simple groups of Ree type, respectively (cf. [16], [21] and [13]). We use the notation G, P, n, e and B_0 as before.

LEMMA 4.1. *Let P be an elementary abelian Sylow 2-subgroup of G of order 8. If $e=21$, then $k(B_0)=8$ and $l(B_0)=5$.*

PROOF. By Lemma 1.14, $N_G(P)/C_G(P)$ is noncyclic of order 21. Hence we can write that $N_G(P)=\langle s, t, C_G(P) \rangle$, $P=\{1, x, x^s, x^{s^2}, z, xz, x^s z, x^{s^2} z\}=\{1, z, z^t, \dots, z^{t^6}\}$ for some $s, t \in N_G(P)$ and involutions $x, z \in P$ with $z^s=z$. Then, by [10, Theorems 68.4 and 65.4], $k(B_0)=l(B_0)+l(b_z)$. Since $e(C_G(z))=3$, $l(b_z)=3$ from Theorem 2.4(1). The calculation of the generalized decomposition matrix of B_0 relative to z is due to J.B. Olsson [18, Theorems 3.15, 3.16 and 3.17]. Let $M=C_G(z)$, $\bar{M}=M/\langle z \rangle$ and $\bar{b}_z=B_0(\bar{M})$. By [10, Theorem 66.3], there is a basic set \bar{W} of \bar{b}_z such that \bar{W} contains the trivial Brauer character and the Cartan matrix of \bar{b}_z with respect to \bar{W} has the form

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then, by [10, Lemma 66.1], there is a basic set W of b_z such that W contains the trivial Brauer character and the Cartan matrix C_z of b_z with respect to W has the form

$$(*) \quad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

We use the following notation here. For an integer $r \geq 0$ and a p -block B , let $E_B(p^r)$ denote the multiplicity of p^r as an elementary divisor of the Cartan matrix of B . If Q is a p -subgroup of a finite group A and if B is a p -block of A , let $n_B(Q)$ denote the multiplicity of Q as a lower defect group of B (cf. [5]. In [5], $n_B(Q)$ is denoted by $m_B^{(1)}(Q)$). By [8, (89.8) Theorem], $E_{B_0}(8)=1$. Since all involutions in G are conjugate, by [5, (7G)], [18, Proposition 1.2] and [10, Theorem 65.4], we get $E_{B_0}(2)=n_{b_z}(\langle z \rangle)$. Since every lower defect group of a 2-block of G contains all 2-subgroups U of G with $U \subseteq Z(G)$, by [5, (7G)], $E_{b_z}(2)=n_{b_z}(\langle z \rangle)$. By (*), $E_{b_z}(2)=2$. Thus $E_{B_0}(2)=2$, so that $l(B_0) \geq 3$. This shows $k(B_0) \geq 6$. Let $\{\chi_i | i=1, \dots, k(B_0)\} = \text{Irr}(B_0)$. Since $l(b_z)=3$, let $N=(n_{i\alpha})_{\substack{1 \leq i \leq k(B_0) \\ 1 \leq \alpha \leq 8}}$ be the matrix of the generalized decomposition numbers of B_0 relative to z with respect to W . Since $|z|=2$, every $n_{i\alpha}$ is an integer. By [4, II (7A) and (4C)], $(n_{i1}, n_{i2}, n_{i3}) \neq (0, 0, 0)$ for every χ_i . For χ_i, χ_j let $a_{ij} = \sum_{1 \leq \alpha, \beta \leq 8} n_{i\alpha} u_{\alpha\beta} n_{j\beta}$, where $C_z^{-1} = (u_{\alpha\beta})_{1 \leq \alpha, \beta \leq 8}$. By Lemma 1.11 and [4, II (7A) and (5G)], all a_{ii} are odd integers. Hence $n_{i1} + n_{i2} + n_{i3}$ is odd for every χ_i . Let N_α be the α -th column of N for each α , and let $N_\alpha N_\beta = \sum_{i=1}^{k(B_0)} n_{i\alpha} n_{i\beta}$ for all α, β . By [10, Theorem 63.3(2)], ${}^t N N = C_z$ where ${}^t N$ is the transposed matrix of N . So $N_\alpha N_\beta = 4$ if $\alpha = \beta$, and $N_\alpha N_\beta = 2$ if $\alpha \neq \beta$. Clearly, $12 = \text{tr}(C_z) = \sum_{i, \alpha} n_{i\alpha}^2$ where $\text{tr}(C_z)$ is the trace of C_z . Then the next three possibilities arise for the nonzero entries of N :

- (i) 2 entries are ± 2 , and 4 entries are ± 1 .
- (ii) 1 entry is ± 2 , and 8 entries are ± 1 .
- (iii) 12 entries are ± 1 .

By elementary calculations as in [18, Theorems 3.15, 3.16 and 3.17] we can write

$$\begin{array}{ccc}
 \delta_1 & 0 & 0 \\
 \delta_2 & 0 & 0 \\
 0 & \delta_3 & 0 \\
 N=0 & \delta_4 & 0 \\
 0 & 0 & \delta_5 \\
 0 & 0 & \delta_6 \\
 \delta_7 & \delta_7 & \delta_7 \\
 \delta_8 & \delta_8 & \delta_8
 \end{array}$$

where $\delta_i = \pm 1$. This shows $k(B_0)=8$, so that $l(B_0)=5$. This completes the proof.

LEMMA 4.2. *Let S be a normal subgroup of G of odd index such that $S \cong J_1 \times (P/(Z_2 \times Z_2 \times Z_2))$ or $S \cong R(q) \times (P/(Z_2 \times Z_2 \times Z_2))$. If $e=21$, then $k(B_0)=2^n$ and $l(B_0)=5$.*

PROOF. We may assume $S=R \times Q$ where $R=J_1$ or $R(q)$ and $Q \cong P/(Z_2 \times Z_2 \times Z_2)$. Let T be a Sylow 2-subgroup of R with $T \times Q = P$. By Lemma 1.12(iii), $N_R(T)/C_R(T)$ is noncyclic of order 21. Hence we can write $N_R(T) = \langle s, t, C_R(T) \rangle$ and $T = \{1, x, x^s, x^{s^2}, z, xz, x^s z, x^{s^2} z\} = \{1, x, x^t, \dots, x^{t^6}\}$ for some $s, t \in N_R(T)$ and for involutions $x, z \in T$ with $z^s = z$. Since $e=21$, $N_G(P) = \langle s, t, C_G(P) \rangle$. We prove the lemma by induction on n . If $n=3$, the lemma is proved from Lemma 4.1 because $P=T$ and P is elementary abelian of order 8 from Lemma 1.12(iii). Suppose $n>3$. Let $Q = \{1=y_1, y_2, \dots, y_{2^{n-3}}\}$. By [10, Lemma 18.5], $\{y_i, zy_i \mid i=1, \dots, 2^{n-3}\}$ is the set of all representatives of G -conjugate classes of P . Then, by [10, Theorems 68.4 and 65.4],

$$k(B_0) = l(B_0) + \sum_{i=2}^{2^{n-3}} l(b_{y_i}) + \sum_{i=1}^{2^{n-3}} l(b_{zy_i}).$$

As in the proof of Lemma 2.2, by induction we get $l(b_{y_i})=5$ for all $i=2, \dots, 2^{n-3}$. We can assume $O(G)=1$ by Lemma 1.1. Since $Q \neq 1$, it follows from Z^* -theorem that $l(B_0)=5$. Since $s \in C_G(zy_i)$ and $t \in C_G(zy_i)$, we have $e(C_G(zy_i))=3$. Hence $l(b_{zy_i})=3$ for all $i=1, \dots, 2^{n-3}$ from Theorem 2.4(1). Thus $k(B_0)=2^n$.

LEMMA 4.3. *Let S be a normal subgroup of G of odd index such that $S \cong J_1 \times (P/(Z_2 \times Z_2 \times Z_2))$. If $e=21$, then $B_0 \cong B_0(S)$.*

PROOF. We can assume $S=J_1 \times Q$ where $Q \cong P/(Z_2 \times Z_2 \times Z_2)$. We use induction on $|G|$. Assume $G \neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index l with $S \subseteq H$. Let $b_0 = B_0(H)$. By induction, $b_0 \cong B_0(S)$. Let s, t, x, z and y_i be the same as in the proof of Lemma 4.2. Since z is an involution in J_1 , by [16, Theorem], $C_{J_1}(z) = A_5 \times \langle z \rangle$ where A_5 is the alternating group of degree 5. Hence $C_S(z) = A_5 \times \langle z \rangle \times Q$. Let $M = C_G(z)$. Clearly $C_S(z) \cong A_5 \times (P/(Z_2 \times Z_2))$ and $C_S(z)$ is a normal subgroup of M of odd index. By the proof of Lemma 4.2, $e(M)=3$. Hence, by Lemma 2.3, we get that $b_z = B_0(M) \cong B_0(A_5 \times (P/Z_2 \times Z_2))$ since $A_5 \cong L_2(5)$. By Lemma 1.16(ii), the Cartan matrix of $B_0(A_5)$ has the form

$$\begin{array}{cccc} & & & 1 \\ & & & 1 \\ 1 & 4 & 2 & 2 \\ & 2 & 2 & 1 \\ & 2 & 1 & 2. \end{array}$$

Thus, by [10, Lemma 66.1], the Cartan matrix C_z of b_z has the form

$$\begin{array}{cccc} & & & 1 \\ & & & 1 \\ 1 & 2^n & 2^{n-1} & 2^{n-1} \\ & 2^{n-1} & 2^{n-1} & 2^{n-2} \\ & 2^{n-1} & 2^{n-2} & 2^{n-1}. \end{array}$$

By Lemma 4.2, $k(B_0)=2^n$. Let $\{\chi_1, \dots, \chi_{2^n}\}=\text{Irr}(B_0)$. We can write $\text{IBr}(b_z)=\{\phi_1^z=1_M, \phi_2^z, \phi_3^z\}$ with $\phi_2^z(1)=\phi_3^z(1)=2$ from Lemma 1.16(i). For each χ_i and ϕ_α^z , let $n_{i\alpha}=d_{i\alpha}^z$ be the generalized decomposition number of B_0 relative to z . Since $|z|=2$, every $n_{i\alpha}$ is an integer. Let $N=(n_{i\alpha})_{\substack{1 \leq i \leq 2^n \\ 1 \leq \alpha \leq 3}}, N_\alpha=(n_{i\alpha})_{1 \leq i \leq 2^n}$ for each α , and $N_\alpha N_\beta = \sum_{i=1}^{2^n} n_{i\alpha} n_{i\beta}$ for each α, β . It follows from [10, Theorems 63.3(2), 63.2 and 65.4] that $N_1 N_1 = 2^n$, $N_2 N_2 = N_3 N_3 = 2^{n-1}$, $N_1 N_2 = N_1 N_3 = 2^{n-1}$ and $N_2 N_3 = 2^{n-2}$. For each χ_i, χ_j , let $a_{ij} = \sum_{1 \leq \alpha, \beta \leq 3} 2^n n_{i\alpha} u_{\alpha\beta} n_{j\beta}$, where $C_z^{-1} = (u_{\alpha\beta})_{1 \leq \alpha, \beta \leq 3}$. Then

$$\begin{aligned} a_{ii} &= 3n_{i1}^2 + 4(n_{i2}^2 + n_{i3}^2) - 4(n_{i1}n_{i2} + n_{i1}n_{i3}) \\ &\equiv n_{i1}^2 \equiv n_{i1} \pmod{2} \end{aligned}$$

for all χ_i . By Lemma 1.11, every χ_i has height zero. Hence, by [4, II (7A) and (5G)], every a_{ii} is odd, so that n_{i1} is odd for all $i=1, \dots, 2^n$. Since $N_1 N_1 = 2^n$, $n_{i1} = \pm 1$ for all $i=1, \dots, 2^n$. Let $\delta_i = n_{i1}$ and $u_i = n_{i2} \delta_i$ for each i . Since $N_1 N_2 = N_2 N_2 = 2^{n-1}$, $\sum_{i=1}^{2^n} u_i = \sum_{i=1}^{2^n} u_i^2$. Thus, $u_i = 1$ or 0 for all $i=1, \dots, 2^n$. Hence exactly 2^{n-1} u_i 's are 1 and the other u_i 's are 0 since $N_1 N_2 = 2^{n-1}$. Then we may assume

$$n_{i2} = \begin{cases} \delta_i & \text{for } i=1, \dots, 2^{n-1} \\ 0 & \text{for } i=2^{n-1}+1, \dots, 2^n. \end{cases}$$

Similarly, exactly 2^{n-1} $(n_{i3} \delta_i)$'s are 1 and the other $(n_{i3} \delta_i)$'s are 0. Since $N_2 N_3 = 2^{n-2}$, we may assume

$$n_{i3} = \begin{cases} \delta_i & \text{for } i=1, \dots, 2^{n-2} \text{ and for } i=2^{n-1}+1, \dots, 3 \cdot 2^{n-2} \\ 0 & \text{for } i=2^{n-2}+1, \dots, 2^{n-1} \text{ and for } i=3 \cdot 2^{n-2}+1, \dots, 2^n. \end{cases}$$

Since $\chi_i(z) = n_{i1} + 2(n_{i2} + n_{i3})$ for each i , we get

$$\chi_i(z) = \begin{cases} \pm 5 & \text{for } i=1, \dots, 2^{n-2} \\ \pm 3 & \text{for } i=2^{n-2}+1, \dots, 3 \cdot 2^{n-2} \\ \pm 1 & \text{for } i=3 \cdot 2^{n-2}+1, \dots, 2^n. \end{cases}$$

Let $C_G(P) = P \times V$. When $G = VH$, $G = C_G(P) \cdot H$, so that $B_0 \cong b_0$ from Proposition 1.6. Thus, we may assume $G \neq VH$. Hence $C_H(P) = P \times V$. Since $b_0 \cong B_0(S)$, it follows from Proposition 1.5 that $|H : VH'| = k'(b_0) = 2^{n-3}$. By [10, Theorem 18.4], $P \cap G' = \{1, x, x^t, \dots, x^{t^6}\}$. Then the order of Sylow 2-subgroups of G' is 8. This implies $2^{n-3} \mid |G : VG'|$ and $2^{n-2} \nmid |G : VG'|$. Thus, by Proposition 1.5, $k'(B_0) = |G : VG'| = l \cdot 2^{n-3}$ where $l = |G : H|$. Since $b_0 \cong B_0(S)$, by Clifford's theorem, Proposition 1.3 and the character table of J_1 [16, p. 148], we get that $\chi_i(z) = 1$ for every $\chi_i \in \text{Irr}(B_0)$ with degree one. These show that the number of

$\chi_i \in \text{Irr}(B_0)$ with $\chi_i(z)=1$ is at least $l \cdot 2^{n-3}$. However, $\chi_i(z)=\pm 1$ only for $i=3 \cdot 2^{n-2}+1, \dots, 2^n$. This is a contradiction since $l \cdot 2^{n-3} > 2^{n-2}$. This completes the proof.

LEMMA 4.4. *Let S be a normal subgroup of G of odd index such that $S \cong R(q) \times (P/(Z_2 \times Z_2 \times Z_2))$. If $e=21$, then $B_0 \cong B_0(S)$.*

PROOF. Let $R=R(q)$. We may assume $S=R \times Q$ where $Q \cong P/(Z_2 \times Z_2 \times Z_2)$. We prove the lemma by induction on $|G|$. Assume $G \neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index l with $S \subseteq H$. Let $b_0=B_0(H)$. By induction, $b_0 \cong B_0(S)$. Let s, t, x, z and y_i be the same as in the proof of Lemma 4.2. Since z is an involution in R , $C_R(z)=L_2(q) \times \langle z \rangle$ from [21, p. 62 III]. (It is noted that we use the notation $R(q)$ as in the sense of [13]). Hence $C_S(z)=L_2(q) \times \langle z \rangle \times Q$. Let $M=C_G(z)$. Then $C_S(z)$ is a normal subgroup of M of odd index and $C_S(z) \cong L_2(q) \times (P/(Z_2 \times Z_2))$. By the proof of Lemma 4.2, $e(M)=3$. Then, by Lemma 2.3, $b_z=B_0(M) \cong B_0(L_2(q) \times (P/(Z_2 \times Z_2)))$. By [21, Theorem (1)], $3 < q \equiv 3 \pmod{8}$, so that as in the proof of Lemma 4.3 the Cartan matrix C_z of b_z has the form

$$\begin{array}{ccc} 2^{n-1} & 2^{n-2} & 2^{n-2} \\ 2^{n-2} & 2^{n-1} & 2^{n-2} \\ 2^{n-2} & 2^{n-2} & 2^{n-1}. \end{array}$$

By Lemma 4.2, $k(B_0)=2^n$. Let $\{\chi_1, \dots, \chi_{2^n}\}=\text{Irr}(B_0)$. We can write $\text{lBr}(b_z)=\{\phi_1^z=1_M, \phi_2^z, \phi_3^z\}$ with $\phi_2^z(1)=\phi_3^z(1)=(q-1)/2$ from Lemma 1.16(i). Let $n_{i\alpha}, N, N_\alpha$ and $N_\alpha N_\beta$ be the same as in the proof of Lemma 4.3. Every $n_{i\alpha}$ is an integer. As in the proof of Lemma 4.3 we get $N_\alpha N_\alpha=2^{n-1}$ for all $\alpha=1, 2, 3$, and $N_\alpha N_\beta=2^{n-2}$ if $\alpha \neq \beta$. Let $C_G(P)=P \times V$. As in the proof of Lemma 4.3 we may assume $G \neq VH$. Since $b_0 \cong B_0(S)$, $k'(b_0)=2^{n-3}$. So that $k'(B_0)=|G:VG'|=l \cdot 2^{n-3}$ as in the proof of Lemma 4.3, where $l=|G:H|$. Since $b_0 \cong B_0(S)$, by [21, p. 74 and pp. 87-88], we can write $\{\tilde{\chi}_{ij} | i=1, \dots, 8; j=1, \dots, 2^{n-3}\}=\text{Irr}(b_0)$ and

	1	z
$\tilde{\chi}_{1j}$	1	1
$\tilde{\chi}_{2j}$	q^2-q+1	-1
$\tilde{\chi}_{3j}$	q^3	q
$\tilde{\chi}_{4j}$	$q(q^2-q+1)$	- q
$\tilde{\chi}_{5j}$	$(q-1)m(q+1+3m)/2$	$-(q-1)/2$
$\tilde{\chi}_{6j}$	$(q-1)m(q+1+3m)/2$	$-(q-1)/2$
$\tilde{\chi}_{7j}$	$(q-1)m(q+1-3m)/2$	$(q-1)/2$
$\tilde{\chi}_{8j}$	$(q-1)m(q+1-3m)/2$	$(q-1)/2$

for $j=1, \dots, 2^{n-3}$, where $q=3^{2^{k+1}}$ and $m=3^k$ for some $k \geq 1$ (cf. [21, Theorem]). By Clifford's theorem, Proposition 1.3 and the above table, we know that if $\chi_i(1)=1$ then $\chi_i(z)=1$. When $n_{i1}=0$, $\chi_i(z)=(n_{i2}+n_{i3})(q-1)/2$. Thus $n_{i1} \neq 0$ if $\chi_i(z) = \pm 1$. Hence the number of $\chi_i \in \text{Irr}(B_0)$ with $n_{i1} \neq 0$ is at least $l \cdot 2^{n-3}$. Since $N_1 N_1 = 2^{n-1}$, we get $l=3$. Fix any χ_i . If $\chi_i|_H = \tilde{\chi}_{2j}$ for some j with $1 \leq j \leq 2^{n-3}$, then $n_{i1} \geq 1$ since $\chi_i(z) = -1$. If $\chi_i|_H = \tilde{\chi}_{2j} + \tilde{\chi}_{2j'} + \tilde{\chi}_{2j''}$ for some j, j', j'' with $1 \leq j < j' < j'' \leq 2^{n-3}$, then $n_{i1} \geq 9$ since $\chi_i(z) = -3$. Let u be the number of $\chi_i \in \text{Irr}(B_0)$ with $\chi_i|_H = \tilde{\chi}_{2j}$, and let v be the number of $\chi_i \in \text{Irr}(B_0)$ with $\chi_i|_H = \tilde{\chi}_{2j} + \tilde{\chi}_{2j'} + \tilde{\chi}_{2j''}$ for $j < j' < j''$. Since $N_1 N_1 = 2^{n-1}$, and since $1 < q^2 - q + 1 < 3(q^2 - q + 1)$, we have

$$2^{n-1} = \sum_{i=1}^{2^n} n_{i1} \geq k'(B_0) + u + 9v = 3 \cdot 2^{n-3} + u + 9v.$$

Then $2^{n-3} \geq u + 9v$. By Proposition 1.4, for every $\tilde{\chi}_{2j}$ there is some χ_i with $(\chi_i|_H, \tilde{\chi}_{2j}) \neq 0$, so that, by Clifford's theorem and Proposition 1.3, $\chi_i|_H = \tilde{\chi}_{2j}$ or $\chi_i|_H = \tilde{\chi}_{2j} + \tilde{\chi}_{2j^g} + \tilde{\chi}_{2j^{g^2}}$ where g is an element of G with $G = \langle g, H \rangle$. By considering the degrees of $\tilde{\chi}_{ij}$, we get that $\tilde{\chi}_{2j^g}$ and $\tilde{\chi}_{2j^{g^2}}$ are both in $\{\tilde{\chi}_{2j'}, |j'=1, \dots, 2^{n-3}\}$. Thus $2^{n-3} \leq u + 3v$, so that $v=0$ and $u=2^{n-3}$. This implies that the number of $\chi_i \in \text{Irr}(B_0)$ with $\chi_i(z) = -1$ is at least 2^{n-3} , so that the number of $\chi_i \in \text{Irr}(B_0)$ with $\chi_i(z) = \pm 1$ is at least 2^{n-1} . Then the number of $\chi_i \in \text{Irr}(B_0)$ with $n_{i1} \neq 0$ is at least 2^{n-1} . Since $N_1 N_1 = 2^{n-1}$, we may assume

$$n_{i1} = \begin{cases} \delta_i & \text{for } i=1, \dots, 2^{n-1} \\ 0 & \text{for } i=2^{n-1}+1, \dots, 2^n \end{cases}$$

where $\delta_i = \pm 1$. Thus $\chi_i(z) = \pm 1$ for all $i=1, \dots, 2^{n-1}$. For all $i=1, \dots, 2^{n-1}$, $\chi_i(z) = \delta_i + (n_{i2} + n_{i3})(q-1)/2$, so that $n_{i2} + n_{i3} = 0$ since $(q-1)/2 \geq 13$. Consequently, $N_1 N_2 + N_1 N_3 = \sum_{i=1}^{2^n} n_{i1} (n_{i2} + n_{i3}) = \sum_{i=1}^{2^{n-1}} \delta_i (n_{i2} + n_{i3}) = 0$. This is a contradiction since $N_1 N_2 = N_1 N_3 = 2^{n-2}$. This completes the proof.

LEMMA 4.5. *Let S be a normal subgroup of G of odd index such that $S \cong L_2(q) \times SL(2, 8)$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$. If $e=21$, then $B_0 \cong B_0(S)$.*

PROOF. Let R_1 and R_2 be Sylow 2-subgroups of $L_2(q)$ and $SL(2, 8)$, respectively. We may assume $S = L_2(q) \times SL(2, 8)$ and $P = R_1 \times R_2$. There are an element $s \in L_2(q)$ and an involution $x \in R_1$ with $R_1 = \{1, x, x^s, x^{s^2}\}$. Similarly, we can write $R_2 = \{1, y, y^t, \dots, y^{t^6}\}$ for some $t \in SL(2, 8)$ and for an involution $y \in R_2$. Since $e=21$, $N_G(P) = \langle s, t, C_G(P) \rangle$ and $N_G(P)/C_G(P)$ is cyclic of order 21. By [10, Lemma 18.5], $\{1, x, y, xy\}$ is the set of all representatives of G -conjugate classes of P . Hence, by [10, Theorems 68.4 and 65.4], $k(B_0) = l(B_0) + l(b_x)$

$+l(b_y)+l(b_{xy})$. Since $s \in C_G(x)$ and $t \in C_G(x)$, we have $e(C_G(x))=7$. Thus $l(b_x)=7$ from Theorem 2.4(1). Similarly, $l(b_y)=3$ from Theorem 2.4(1). For integers i and j , $(xy)^{s^i t^j} = xy$ if and only if $3|i$ and $7|j$. This implies $N_M(P)=C_M(P)$ where $M=C_G(xy)$. Thus, by [10, Theorem 18.7 and Corollary 65.3], $l(b_{xy})=l(B_0(M))=1$. Since G is nonsolvable, $l(B_0) \geq 2$ from [10, Corollary 65.3], so that $k(B_0) \geq 13$.

Now, we prove the lemma by induction on $|G|$. Assume $G \neq S$. By [12, Theorem], G has a normal subgroup H of odd prime index l with $S \subseteq H$. Let $b_0 = B_0(H)$. We know $b_0 \cong B_0(S)$ by induction. From the character tables of $L_2(q)$ and $SL(2, 8)$ (cf. [10, Theorems 38.1 and 38.2]), we can write

$$\begin{array}{ccc} & 1 & x \\ \theta_1 & 1 & 1 \\ \theta_2 & (q+\varepsilon)/2 & -\varepsilon \\ \theta_3 & (q+\varepsilon)/2 & -\varepsilon \\ \theta_4 & q & \varepsilon \end{array} \quad \varepsilon = \begin{cases} -1 & \text{if } q \equiv 3 \pmod{8} \\ 1 & \text{if } q \equiv 5 \pmod{8} \end{cases}$$

where $\{\theta_1, \theta_2, \theta_3, \theta_4\} = \text{Irr}(B_0(L_2(q)))$ and

$$\begin{array}{ccc} & 1 & y \\ \zeta_1 & 1 & 1 \\ \zeta_j & 7 & -1 \quad \text{for } j=2, 3, 4, 5 \\ \zeta_j & 9 & 1 \quad \text{for } j=6, 7, 8 \end{array}$$

where $\{\zeta_1, \dots, \zeta_8\} = \text{Irr}(B_0(SL(2, 8)))$. Since $b_0 \cong B_0(S)$, we may write $\text{Irr}(b_0) = \{\tilde{\chi}_{ij} | i=1, \dots, 4; j=1, \dots, 8\}$ with $\tilde{\chi}_{ij}|_S = \theta_i \zeta_j$ for all i, j . Hence the degrees of all $\tilde{\chi}_{ij}$ are 1, 7, 9, $(q+\varepsilon)/2$, $7(q+\varepsilon)/2$, $9(q+\varepsilon)/2$, q , $7q$ and $9q$. Next, we want to show that

$$\begin{aligned} (*) \quad & \text{if } \tilde{\chi}, \tilde{\chi}' \in \text{Irr}(b_0) \text{ with } \tilde{\chi}(1) = \tilde{\chi}'(1) \\ & \text{then } \tilde{\chi}(xy) = \tilde{\chi}'(xy) = \pm 1. \end{aligned}$$

Case 1. $\varepsilon=1$: Clearly $\{1, 9, 7(q+1)/2, 9, 9q\} \cap \{7, (q+1)/2, 9(q+1)/2, 7q\} = \emptyset$. Hence, by considering the values $\tilde{\chi}_{ij}(1)$ and $\tilde{\chi}_{ij}(xy)$, we get (*).

Case 2. $\varepsilon=-1$: Since $\{1, 9, (q-1)/2, 9(q-1)/2, 7q\} \cap \{7, 7(q-1)/2, q, 9q\} = \emptyset$, we obtain (*) as in Case 1.

We get from Clifford's theorem, Proposition 1.3, (*) and the above character tables of $L_2(q)$ and $SL(2, 8)$ that $\chi(xy) = \pm 1$ or $\pm l$ for every $\chi \in \text{Irr}(B_0)$. Let $k=k(B_0)$, and let m be the number of $\chi \in \text{Irr}(B_0)$ with $\chi(xy) = \pm 1$. Hence we can write $\text{Irr}(B_0) = \{\chi_1=1_G, \chi_2, \dots, \chi_m, \chi_{m+1}, \dots, \chi_k\}$ such that

$$\chi_i(xy) = \begin{cases} \pm 1 & \text{for } i=1, \dots, m \\ \pm l & \text{for } i=m+1, \dots, k. \end{cases}$$

Since $l(b_{xy})=1$, as in the proof of Lemma 1.9,

$$(**) \quad 32 = \sum_{i=1}^k \chi_i(xy)^2 = m + (k-m)l^2.$$

Firstly, suppose $k=m$. Then $\chi_i(xy)=\pm 1$ for all $\chi_i \in \text{Irr}(B_0)$. Since $k=m=32$ and since $b_0 \cong B_0(S)$, we have $k(B_0)=k(b_0)=32$. Hence $l(B_0)=21$, so that $l(B_0)=l(b_0)$ since $b_0 \cong B_0(S)$. Thus, by (*) and Corollary 1.8, $B_0 \cong b_0$. Thus, we may assume $k > m$. Since $k \geq 13$, by (**), $l=3$. So that $k-m=1$ or 2. Let $C_G(P) = P \times V$. Since $k > m$ and $b_0 \cong B_0(S)$, we know $B_0 \not\cong b_0$. Hence $G \neq VH$ from Proposition 1.6. This shows $C_H(P) = P \times V$. Thus, by Proposition 1.5, $|H:VH'| = k'(b_0)=1$ since $b_0 \cong B_0(S)$. Then $H=VH'$. Since G/H is cyclic, $VG'=VH'=H$. Hence $k'(B_0)=|G:VG'|=l=3$ by Proposition 1.5. So that we may assume that $\chi_1(1)=\chi_2(1)=\chi_3(1)=1$ and $\chi_i(1)>1$ for all $i=4, \dots, k$.

Case A. $k-m=1$: By (**), we get $m=23$ and $k=24$. Then $\chi_i(xy)=\pm 1$ for $i=1, \dots, 23$ and $\chi_{24}(xy)=\pm 3$. Since $b_0 \cong B_0(S)$, by Clifford's theorem and Proposition 1.3, $\chi_i|_H = 1_H$ for $i=1, 2, 3$, $\chi_i|_H \neq 1_H$ and $\chi_i|_H \in \text{Irr}(b_0)$ for $i=4, \dots, 23$, $\chi_{24}|_H = \tilde{\chi}_{i_1} + \tilde{\chi}_{i_2} + \tilde{\chi}_{i_3}$ where $\tilde{\chi}_{i_1}$, $\tilde{\chi}_{i_2}$ and $\tilde{\chi}_{i_3}$ are distinct G -conjugate elements in $\text{Irr}(b_0)$. On the other hand, it follows from Proposition 1.4 that for every $\tilde{\chi} \in \text{Irr}(b_0)$ there is some $\chi_i \in \text{Irr}(B_0)$ with $(\chi_i|_H, \tilde{\chi}) \neq 0$. These show $k(b_0) \leq 1+20+3=24$. But $k(b_0)=32$ since $b_0 \cong B_0(S)$. Then we have a contradiction.

Case B. $k-m=2$: We have from (**) that $\chi_i(xy)=\pm 1$ for $i=1, \dots, 14$, $\chi_{15}(xy)=\pm 3$ and $\chi_{16}(xy)=\pm 3$. Hence as in Case A we get $k(b_0) \leq 1+11+6=18$. This is a contradiction as in Case A. This completes the proof.

LEMMA 4.6. *Let S be a normal subgroup of G of odd index such that $S \cong L_2(q) \times SL(2, 8) \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$. If $e=21$, then $k(B_0)=2^n$ and $l(B_0)=21$.*

PROOF. If $n=5$, we can prove the lemma by Lemma 4.5 (cf. Lemma 1.12 and Theorem 2.4). If $n > 5$, we can verify the lemma by induction on n as in the proof of Lemma 4.2.

LEMMA 4.7. *Assume as in Lemma 4.6. Then $B_0 \cong B_0(S)$.*

PROOF. We may assume $S = L_2(q) \times SL(2, 8) \times Q$ with $Q \cong P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2)$. We use induction on $|G|$ as before. Assume $G \neq S$. Hence G has a normal subgroup H of odd prime index with $S \subseteq H$ from [12, Theorem]. Let $b_0 = B_0(H)$. By induction, $b_0 \cong B_0(S)$. Let x and y be involutions in $L_2(q)$ and

$SL(2, 8)$, respectively. As in the proof of Lemma 4.2, $l(b_{xy})=1$. By Lemma 4.6, $k(B_0)=2^n$. Thus, $\chi(xy)=\pm 1$ for all $\chi \in \text{Irr}(B_0)$ from Lemma 1.9. By Lemma 4.6, $k(B_0)=k(b_0)$ and $l(B_0)=l(b_0)$. Since $b_0 \cong B_0(S)$, as in the proof of Lemma 4.5, we get that if $\tilde{\chi}, \tilde{\chi}' \in \text{Irr}(b_0)$ with $\tilde{\chi}(1)=\tilde{\chi}'(1)$ then $\tilde{\chi}(xy)=\tilde{\chi}'(xy)=\pm 1$. These imply $B_0 \cong b_0$ from Corollary 1.8. This completes the proof.

Next, we state the following main result of this section. That is proved by making use of Lemmas 4.2-4.7.

THEOREM 4.8. *Let P be an abelian Sylow 2-subgroup of G , and let $S=O'(G/O(G))$. If $e=e(S)=21$, then we have the following.*

(1) $k(B_0)=|P|$ and

$$l(B_0)=\begin{cases} 5 & \text{if } N_G(P)/C_G(P) \text{ is noncyclic} \\ 21 & \text{if } N_G(P)/C_G(P) \text{ is cyclic.} \end{cases}$$

(2) One of the following holds:

(i) $B_0 \cong B_0(J_1 \times (P/(Z_2 \times Z_2 \times Z_2)))$,

(ii) $B_0 \cong B_0(R(q) \times (P/(Z_2 \times Z_2 \times Z_2)))$,

(iii) $B_0 \cong B_0(L_2(q) \times SL(2, 8) \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2)))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$.

PROOF. By Lemma 1.1, we may assume $O(G)=1$. By Proposition 1.10 and Lemma 1.12, one of the following holds:

(i) $S \cong J_1 \times (P/(Z_2 \times Z_2 \times Z_2))$,

(ii) $S \cong R(q) \times (P/(Z_2 \times Z_2 \times Z_2))$,

(iii) $S \cong L_2(q) \times SL(2, 8) \times (P/(Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$. Then we can prove the theorem by Lemmas 4.2-4.7.

5. The case when P is elementary abelian of order 8

In this section we consider the case when G has elementary abelian Sylow 2-subgroups of order 8. In particular, we shall determine B_0 in the case when G is nonsolvable, $e=21$ and $e(S) \neq 21$ where $S=O'(G/O(G))$. Throughout this section we assume that G has an elementary abelian Sylow 2-subgroup P of order 8 and we use the notation e and B_0 as before.

By Lemma 1.14 and Remark 1 of §1, it is sufficient to consider the cases when $e=3, 7$ and 21 .

PROPOSITION 5.1. (i) If $e=3$, then $k(B_0)=8$ and $l(B_0)=3$.

(ii) If $e=7$, then $k(B_0)=8$ and $l(B_0)=7$.

(iii) If $e=21$, then $k(B_0)=8$ and $l(B_0)=5$.

PROOF. (i) We can write $N_G(P)=\langle s, C_G(P) \rangle$ for some $s \in N_G(P)$. There is an involution $x \in P$ with $x^s \neq x$. Hence $l(b_x)=1$ as in the proof of Lemma 2.1. Then $k(B_0)=8$ from Lemma 1.15(1). On the other hand, $l(B_0)=3$ by Theorem 2.4(1).

(ii) We can verify (ii) as in (i).

(iii) We have already proved (iii) in Lemma 4.1.

PROPOSITION 5.2. *There is a basic set W of B_0 such that W contains the trivial Brauer character and the decomposition matrix of B_0 with respect to W has the form*

$$\begin{array}{ccccccccc}
 1_G & 1 & 0 & 0 & & 1 & & & & 1 & 0 & 0 & 0 & 0 \\
 & \delta_2 & 0 & 0 & & \delta_2 & & & & 0 & \delta_2 & 0 & 0 & 0 \\
 & 0 & \delta_3 & 0 & & \delta_3 & & 0 & & 0 & 0 & \delta_3 & 0 & 0 \\
 & 0 & \delta_4 & 0 & & & & & & 0 & 0 & 0 & \delta_4 & 0 \\
 & 0 & 0 & \delta_5 & & 0 & & & & 0 & 0 & 0 & 0 & \delta_5 \\
 & 0 & 0 & \delta_6 & & & & & & \delta_6 & 0 & 0 & \delta_6 & \delta_6 \\
 & \delta_7 & \delta_7 & \delta_7 & & & & & \delta_7 & 0 & \delta_7 & 0 & \delta_7 & \delta_7 \\
 & \delta_8 & \delta_8 & \delta_8 & & \delta_8 & \delta_8 & \delta_8 & \cdots & \delta_8 & 0 & 0 & \delta_8 & \delta_8 & \delta_8 \\
 e=3 & & & & & e=7 & & & & e=21 & & & & &
 \end{array}$$

where $\delta_i = \pm 1$.

PROOF. Case 1. $e=3$: Clear from Proposition 5.1(i) and the proof of Lemma 4.1.

Case 2. $e=7$: By Proposition 5.1(ii), $k(B_0)=8$. Let $\{\chi_1, \dots, \chi_8\} = \text{Irr}(B_0)$. By the proof of Proposition 5.1(ii), G has an involution x with $l(b_x)=1$. By Lemma 1.9, we get $\chi_i(x) = \pm 1$ for all i . On the other hand, $\sum_{i=1}^8 \chi_i(x) \chi_i = 0$ on 2'-elements of G from [10, Theorem 63.3(1)]. Thus, the assertion is proved.

Case 3. $e=21$: Let z be an involution in G . By the proof of Lemma 4.1, the generalized decomposition matrix of B_0 relative to z with respect to some basic set of b_z has the same form as in Case 1. Hence, by [10, Theorem 63.3(1)], we can verify the proposition.

LEMMA 5.3. *Assume $e=21$, $O(G)=1$, $O'(G)=SL(2, 8)$ and G has a normal subgroup H of odd prime index with $e(H)=7$. Then for any involution z in G we get*

$$\chi_i(1) = \begin{cases} 1 & \text{for } i=1, 2, 3 \\ 7 & \text{for } i=4, 5, 6 \\ 21 & \text{for } i=7 \\ 27 & \text{for } i=8, \end{cases} \quad \chi_i(z) = \begin{cases} 1 & \text{for } i=1, 2, 3 \\ -1 & \text{for } i=4, 5, 6 \\ -3 & \text{for } i=7 \\ 3 & \text{for } i=8 \end{cases}$$

where $\{\chi_1=1_G, \chi_2, \dots, \chi_8\} = \text{Irr}(B_0)$.

PROOF. Let $S=O'(G)=SL(2, 8)$. By Lemmas 1.12 and 1.14, we can write $N_S(P)=\langle s, C_S(P) \rangle$, $N_H(P)=\langle s, C_H(P) \rangle$ and $N_G(P)=\langle s, t, C_G(P) \rangle$ for some $s \in N_S(P)$ and $t \in N_G(P)$ such that s and t have orders 7 and 3 modulo $C_G(P)$, respectively. Clearly, $G/H=\langle tH \rangle$. Let $b_0=B_0(H)$, and let $C_G(P)=P \times V$. By Proposition 5.2, $B_0 \cong b_0$. Hence $VH=H$ from Proposition 1.6. Then $C_G(P)=C_H(P)$ and $|G:H|=3$. We may assume $z \in P$. Let $M=C_G(z)$. By the proof of Lemma 2.1, $C_S(z)$ is a 2-nilpotent normal subgroup of M , so that M is solvable. By the proof of Lemma 4.1, $e(M)=3$. Thus, by Lemma 1.1, $B_0(M) \cong B_0(P \cdot Z_3)$ where $P \cdot Z_3$ is the semi-direct product of its normal subgroup P by Z_3 and it is not the direct product $P \times Z_3$. Thus, as in the proof of Lemma 4.1 we know the generalized decomposition numbers of B_0 relative to z . So we can write

$$(*) \quad \chi_i(z) = \begin{cases} \pm 1 & \text{for } i=1, \dots, 6 \\ \pm 3 & \text{for } i=7, 8 \end{cases}$$

for suitable indexing of χ_2, \dots, χ_8 . By Lemma 2.1, $b_0 \cong B_0(S)$. Hence, by [10, Theorem 38.2],

$$(**) \quad \tilde{\chi}_i(1) = \begin{cases} 1 & \text{for } i=1 \\ 7 & \text{for } i=2, \dots, 5 \\ 9 & \text{for } i=6, 7, 8 \end{cases} \quad \tilde{\chi}_i(z) = \begin{cases} 1 & \text{for } i=1 \\ -1 & \text{for } i=2, \dots, 5 \\ 1 & \text{for } i=6, 7, 8 \end{cases}$$

where $\{\tilde{\chi}_1, \dots, \tilde{\chi}_8\} = \text{Irr}(b_0)$. Since $|G:VH|=|G:H|=3$, we get $3 \mid |G:VG'|$. By Proposition 1.5, $k'(B_0)=|G:VG'|$. By (**), $k'(b_0)=1$, so that $|G:VG'|=3$ from Frobenius reciprocity. So we may assume that $\chi_1|_H=\chi_2|_H=\chi_3|_H=\tilde{\chi}_1$ from (*), (**) and Proposition 1.3. Similarly, we may also assume that $\chi_7|_H=\tilde{\chi}_3+\tilde{\chi}_4+\tilde{\chi}_5$ and $\chi_8|_H=\tilde{\chi}_6+\tilde{\chi}_7+\tilde{\chi}_8$. Then we get $\chi_4|_H=\chi_5|_H=\chi_6|_H=\tilde{\chi}_2$. This completes the proof.

The next theorem is the main result of this section.

THEOREM 5.4. Let $\bar{G}=G/O(G)$ and $S=O'(\bar{G})$. If G is nonsolvable, $e=21$ and $e(S) \neq 21$, then we have the following.

(i) $S \cong SL(2, 8)$.

(ii) For any subnormal subgroup \bar{L} of \bar{G} of odd index with $e(\bar{L})=21$,

$$B_0 \cong B_0(\bar{L}).$$

PROOF. We may assume $O(G)=1$ by Lemma 1.1, so that $S=O'(G)$.

(i) Noncyclic groups of order 21 have no normal subgroups of order 3. Thus, by Lemma 1.14, $e(S)=7$. Then $S \cong SL(2, 8)$ from Proposition 1.10 and Lemma 1.12.

(ii) Firstly, we want to show that

$$(*) \quad \begin{cases} \text{if } L \text{ is a normal subgroup of } G \text{ such that } |G:L| \text{ is an odd prime and} \\ e(L)=21 \text{ and if } H \text{ is a normal subgroup of } L \text{ such that } |L:H| \text{ is an} \\ \text{odd prime and } e(H)=7, \text{ then } B_0 \cong B_0(L). \end{cases}$$

Let $b_0=B_0(L)$, and let z be an involution in G . By Lemma 5.3, we get

$$(**) \quad \tilde{\chi}_i(1) = \begin{cases} 1 & \text{for } i=1, 2, 3 \\ 7 & \text{for } i=4, 5, 6 \\ 21 & \text{for } i=7 \\ 27 & \text{for } i=8, \end{cases} \quad \tilde{\chi}_i(z) = \begin{cases} 1 & \text{for } i=1, 2, 3 \\ -1 & \text{for } i=4, 5, 6 \\ -3 & \text{for } i=7 \\ 3 & \text{for } i=8 \end{cases}$$

where $\{\tilde{\chi}_1, \dots, \tilde{\chi}_8\} = \text{Irr}(b_0)$. As in the proof of Lemma 5.3, using the generalized decomposition numbers of B_0 relative to z ,

$$(***) \quad \chi_i(z) = \begin{cases} \pm 1 & \text{for } i=1, \dots, 6 \\ \pm 3 & \text{for } i=7, 8 \end{cases}$$

where $\{\chi_1, \dots, \chi_8\} = \text{Irr}(B_0)$. Since $|G:L|$ is an odd prime, $I_G(\tilde{\chi}_7) = I_G(\tilde{\chi}_8) = G$ from (**). Thus, by Proposition 1.4, Clifford's theorem, (**) and (***), we may assume that $\chi_7|_L = \tilde{\chi}_7$ and $\chi_8|_L = \tilde{\chi}_8$. By Clifford's theorem, Proposition 1.3, (***), and (**), we have $\chi_i|_L \in \text{Irr}(b_0)$ for $i=1, \dots, 6$. Thus, by Proposition 1.4, we may assume that $\chi_i|_L = \tilde{\chi}_i$ for $i=1, \dots, 6$. These show $I_G(\tilde{\chi}_j) = G$ for all $\tilde{\chi}_j \in \text{Irr}(b_0)$. By Proposition 5.1(3), $k(B_0) = k(b_0)$ and $l(B_0) = l(b_0)$. Thus, $B_0 \cong b_0$ from Corollary 1.7. Then, (*) is proved. Since G/S is solvable by [12, Theorem], by repeating the above way, we can prove (ii).

REMARK 1. If G is solvable, we easily know B_0 since we may assume $O(G)=1$ from Lemma 1.1. Assume G is nonsolvable. If $e=3$ or 7, we know B_0 from Theorem 2.4. If $e=21$, we know B_0 from Theorems 4.8 and 5.4.

REMARK 2. By Remark 1 of §2, there is a finite group G with elementary abelian Sylow 2-subgroups of order 8 such that $e(G)=21$ and $e(S)=7$ where $S=O'(G/O(G))$.

6. The case when P is elementary abelian of order 16

In this section we deal with the case when G has elementary abelian Sylow 2-subgroups of order 16. Specially, we are interested in the case where e is not prime. When e is 9 or 21, the similar phenomenon to Theorem 5.4 occurs. Throughout this section we assume that G has an elementary abelian Sylow 2-subgroup P of order 16 and we use the notation e and B_0 as usual.

By Lemma 1.13 and Remark 1 of §1, it is enough to consider the cases when $e=3, 5, 7, 9, 15$ and 21 .

PROPOSITION 6.1. *If G is solvable and $e=3$, then one of the following holds.*

(i) $B_0 \cong B_0(M)$ where M is a semi-direct product of its normal subgroup P by $\langle t \rangle$ such that $P = \langle x, y, z, w \rangle$ is elementary abelian of order 16, $\langle t \rangle$ is cyclic of order 3, $x^t = y$, $y^t = xy$, $z^t = w$ and $w^t = zw$. In this case $k(B_0) = 8$.

(ii) $B_0 \cong B_0(L)$ where L is a semi-direct product of its normal subgroup P by $\langle t \rangle$ such that $P = \langle x, y, z, w \rangle$ is elementary abelian of order 16, $\langle t \rangle$ is cyclic of order 3, $x^t = x$, $y^t = y$, $z^t = w$ and $w^t = zw$. In this case $k(B_0) = 16$.

PROOF. By Lemma 1.1, we may assume $O(G) = 1$. Hence G is a semi-direct product of its normal subgroup P by Z_3 and G is not the direct product $P \times Z_3$. Let $G = P \langle t \rangle$ where $\langle t \rangle$ is cyclic of order 3, and let $P = \langle x, y, z, w \rangle$. We may assume that

(i) $x^t = y$, $y^t = xy$, $z^t = w$, $w^t = zw$

or

(ii) $x^t = x$, $y^t = y$, $z^t = w$, $w^t = zw$.

Then we can easily prove the assertion.

PROPOSITION 6.2. *Let D be the decomposition matrix of B_0 . If $e=3$, then we have the following.*

(i) *When G is solvable, D has the form*

$$\begin{array}{ccc}
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1 \\
 1 & 1 & 1 \\
 1 & 1 & 1 \\
 1 & 1 & 1 \\
 1 & 1 & 1 \\
 1 & 1 & 1, \text{ or} \\
 k(B_0) = 8 & &
 \end{array}
 \begin{array}{ccc}
 1 & 0 & 0 \\
 1 & 0 & 0 \\
 1 & 0 & 0 \\
 1 & 0 & 0 \\
 0 & 1 & 0 \\
 0 & 1 & 0 \\
 0 & 1 & 0 \\
 0 & 1 & 0 \\
 0 & 0 & 1
 \end{array}$$

$$\begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1. \\
k(B_0)=16
\end{array}$$

(ii) When G is nonsolvable, we obtain D from Theorem 2.4(2) and Lemma 1.16(ii).

PROOF. The assertion is proved by Proposition 6.1.

PROPOSITION 6.3. *If $e=5$, then G is solvable, $B_0 \cong B_0(P \cdot Z_5)$ where $P \cdot Z_5$ is the semi-direct product of its normal subgroup P by Z_5 and it is not the direct product $P \times Z_5$, and the decomposition matrix of B_0 has the form*

$$\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1.
\end{array}$$

PROOF. By Proposition 1.10 and Lemma 1.12, G is solvable since we may assume $O(G)=1$ by Lemma 1.1. Hence G is the semi-direct product of P by Z_5 , and it is not the direct product $P \times Z_5$. The decomposition matrix of B_0 is easily obtained.

PROPOSITION 6.4. *If $e=7$, then there is a basic set W of B_0 such that W contains the trivial Brauer character and the decomposition matrix of B_0 with respect to W has the form*

$$\begin{array}{ccccccc}
& & 1_G & & & & \\
1_G & & 1 & & & & \\
& \delta_2 & & \mathbf{0} & & & \\
& & \delta_3 & & & & \\
& & \delta_4 & \cdot & & & \\
& & & \cdot & \cdot & & \\
& \mathbf{0} & & & \cdot & \cdot & \\
& & & & & \delta_{13} & \\
& & & & & \delta_{14} & \\
\delta_{15} & \delta_{15} & \cdot & \cdot & \cdot & \delta_{15} & \\
\delta_{16} & \delta_{16} & \cdot & \cdot & \cdot & \delta_{16} &
\end{array}$$

PROOF. As in the proof of Lemma 4.1 we can prove the assertion by Proposition 5.2.

(1) If G has an involution x with $b_x \cong B_0(P \cdot Z_3)$ where $P \cdot Z_3$ is a semi-direct product of P by Z_3 and it is not the direct product $P \times Z_3$, then the generalized decomposition matrix D^x of B_0 relative to x has the form (*).

(2) If G has an involution x with $b_x \cong B_0(Z_2 \times Z_2 \times L_2(q))$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$, then the generalized decomposition matrix D^x of B_0 relative to x is as follows:

- (i) When $3 < q \equiv 3 \pmod{8}$, D^x has the form (*).
- (ii) When $3 < q \equiv 5 \pmod{8}$, D^x has the form (**).

$$\begin{array}{cccc}
& \mathbf{1}_M & & \\
\mathbf{1}_G & 1 & 0 & 0 \\
& \tilde{\partial}_2 & 0 & 0 \\
& \tilde{\partial}_3 & 0 & 0 \\
& \tilde{\partial}_4 & 0 & 0 \\
& 0 & \tilde{\partial}_5 & 0 \\
& 0 & \tilde{\partial}_6 & 0 \\
& 0 & \tilde{\partial}_7 & 0 \\
& 0 & \tilde{\partial}_8 & 0 \\
& 0 & 0 & \tilde{\partial}_9 \\
& 0 & 0 & \tilde{\partial}_{10} \\
& 0 & 0 & \tilde{\partial}_{11} \\
& 0 & 0 & \tilde{\partial}_{12} \\
& \tilde{\partial}_{13} & \tilde{\partial}_{13} & \tilde{\partial}_{13} \\
& \tilde{\partial}_{14} & \tilde{\partial}_{14} & \tilde{\partial}_{14}
\end{array}
\qquad
\begin{array}{cccc}
& \mathbf{1}_M & & \\
\mathbf{1}_G & 1 & 0 & 0 \\
& \tilde{\partial}_2 & 0 & 0 \\
& \tilde{\partial}_3 & 0 & 0 \\
& \tilde{\partial}_4 & 0 & 0 \\
& \tilde{\partial}_5 & \tilde{\partial}_5 & 0 \\
& \tilde{\partial}_6 & \tilde{\partial}_6 & 0 \\
& \tilde{\partial}_7 & \tilde{\partial}_7 & 0 \\
& \tilde{\partial}_8 & \tilde{\partial}_8 & 0 \\
& \tilde{\partial}_9 & 0 & \tilde{\partial}_9 \\
& \tilde{\partial}_{10} & 0 & \tilde{\partial}_{10} \\
& \tilde{\partial}_{11} & 0 & \tilde{\partial}_{11} \\
& \tilde{\partial}_{12} & 0 & \tilde{\partial}_{12} \\
& \tilde{\partial}_{13} & \tilde{\partial}_{13} & \tilde{\partial}_{13} \\
& \tilde{\partial}_{14} & \tilde{\partial}_{14} & \tilde{\partial}_{14}
\end{array}$$

$$\begin{array}{ccc}
\delta_{15} & \delta_{15} & \delta_{15} \\
\delta_{16} & \delta_{16} & \delta_{16} \\
(*) & & (**)
\end{array}$$

where $\delta_i = \pm 1$ and $M = C_G(x)$.

PROOF. (1) By Proposition 6.2(i), we know the Cartan matrix of b_x . Hence the assertion is proved as in the proof of Lemma 4.1.

(2) We obtain the Cartan matrix of b_x from Lemma 1.16(ii). Thus we can verify (2) as in the proof of (1).

LEMMA 6.6. Assume $e=9$, $O(G)=1$, $O'(G)=Z_2 \times Z_2 \times L_2(q)$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$, and G has a normal subgroup H of odd prime index with $e(H)=3$. Let $b_0 = B_0(H)$, and let x and z be involutions in $Z(O'(G))$ and $L_2(q)$, respectively. Then we have the following.

(i) $\chi_i|_H = \chi_{i+1}|_H = \chi_{i+2}|_H = \tilde{\chi}_i$ for $i=1, 5, 9, 13$

$\chi_j|_H = \tilde{\chi}_{j-2} + \tilde{\chi}_{j-1} + \tilde{\chi}_j$ for $j=4, 8, 12, 16$

and the values $\tilde{\chi}_i(1)$, $\tilde{\chi}_i(x)$, $\tilde{\chi}_i(z)$ and $\tilde{\chi}_i(xz)$ are as follows:

	1	x	z	xz
$\tilde{\chi}_1, \tilde{\chi}_2$	1	1	1	1
$\tilde{\chi}_3, \tilde{\chi}_4$	1	-1	1	-1
$\tilde{\chi}_5, \tilde{\chi}_6$	$(q+\varepsilon)/2$	$(q+\varepsilon)/2$	$-\varepsilon$	$-\varepsilon$
$\tilde{\chi}_7, \tilde{\chi}_8$	$(q+\varepsilon)/2$	$-(q+\varepsilon)/2$	$-\varepsilon$	ε
$\tilde{\chi}_9, \tilde{\chi}_{10}$	$(q+\varepsilon)/2$	$(q+\varepsilon)/2$	$-\varepsilon$	$-\varepsilon$
$\tilde{\chi}_{11}, \tilde{\chi}_{12}$	$(q+\varepsilon)/2$	$-(q+\varepsilon)/2$	$-\varepsilon$	ε
$\tilde{\chi}_{13}, \tilde{\chi}_{14}$	q	q	ε	ε
$\tilde{\chi}_{15}, \tilde{\chi}_{16}$	q	$-q$	ε	$-\varepsilon$

where $\{\chi_1=1_G, \chi_2, \dots, \chi_{16}\} = \text{Irr}(B_0)$, $\{\tilde{\chi}_1=1_H, \tilde{\chi}_2, \dots, \tilde{\chi}_{16}\} = \text{Irr}(b_0)$ and $\varepsilon = -1$ if $q \equiv 3 \pmod{8}$; $\varepsilon = 1$ if $q \equiv 5 \pmod{8}$.

(ii) $\phi_i|_H = \phi_{i+1}|_H = \phi_{i+2}|_H = \tilde{\phi}_{(i+2)/3}$ for $i=1, 4, 7$

$\tilde{\phi}_{j/3}^g = \phi_{j-2} + \phi_{j-1} + \phi_j$ for $j=3, 6, 9$

where $\{\phi_1=1_G, \phi_2, \dots, \phi_9\} = \text{IBr}(B_0)$ and $\{\tilde{\phi}_1=1_H, \tilde{\phi}_2, \tilde{\phi}_3\} = \text{IBr}(b_0)$.

PROOF. Let $S = O'(G) = \langle x, y \rangle \times L_2(q)$ and $P = \langle x, y, z, w \rangle$ where $\langle z, w \rangle$ is a 2-Sylow subgroup of $L_2(q)$. We can write $N_S(P) = \langle s, C_S(P) \rangle$ for some $s \in N_S(P)$. We may assume $z^s = w$ and $w^s = zw$. We can also write $N_G(P) = \langle s, t, C_G(P) \rangle$ for some $t \in N_G(P)$ where s and t have order 3 modulo $C_G(P)$ since $e=9$ (cf. Lemma 1.13). We may assume $x^t = y$, $y^t = xy$, $z^t = z$ and $w^t = w$. As in the proof of Lemma 5.3, we get $G/H = \langle tH \rangle$, $C_G(P) = C_H(P)$ and $|G:H|=3$. By [10, Lemma

18.5], $\{1, x, z, xz\}$ is the set of all representatives of G -conjugate classes of P . As before, $l(b_x)=l(b_z)=3$ and $l(b_{xz})=1$. By Lemma 3.1, $k(B_0)=16$. Since S is normal in $C_G(x)$ and $e(C_G(x))=3$, it follows from Lemma 2.3 that $b_x \cong B_0(Z_2 \times Z_2 \times L_2(q))$. Thus, by Lemmas 1.16(i) and 6.5(2), we may assume

$$(*) \quad \chi_i(x) = \begin{cases} \pm 1 & \text{for } i=1, \dots, 4 \\ \pm(q+\varepsilon)/2 & \text{for } i=5, \dots, 12 \\ \pm q & \text{for } i=13, \dots, 16. \end{cases}$$

Since $e(H)=3$, by Lemma 2.3, $b_0 \cong B_0(S)$. Let $C_G(P)=P \times V$. By [10, Theorem 18.4], $P \cap G' = P$, so that $|G:VG'|$ is odd. Since $b_0 \cong B_0(S)$ and since $C_H(P)=P \times V$, by Proposition 1.5, $|H:VH'|=4$. Thus, $|G:VG'|=3$, so that $k'(B_0)=3$ from Proposition 1.5. Since $b_0 \cong B_0(S)$, by [10, Theorem 38.1], we know the values of $\tilde{\chi}_i|_S$ for all i . Then we get the table in (i). Using this we may assume that

$$(**) \quad \begin{aligned} I_G(\tilde{\chi}_i) &= G \quad \text{for } i=1, 5, 9, 13 \\ I_G(\tilde{\chi}_{j-2}) &= I_G(\tilde{\chi}_{j-1}) = I_G(\tilde{\chi}_j) = H \\ \tilde{\chi}_{j-2}^t &= \tilde{\chi}_{j-1}, \tilde{\chi}_{j-1}^t = \tilde{\chi}_j \end{aligned} \quad \left. \vphantom{\begin{aligned} I_G(\tilde{\chi}_{j-2}) &= I_G(\tilde{\chi}_{j-1}) = I_G(\tilde{\chi}_j) = H \\ \tilde{\chi}_{j-2}^t &= \tilde{\chi}_{j-1}, \tilde{\chi}_{j-1}^t = \tilde{\chi}_j \end{aligned}} \right\} \quad \text{for } j=4, 8, 12, 16.$$

By (*) and (**), we may assume that $\chi_1|_H = \chi_2|_H = \chi_3|_H = \tilde{\chi}_1$. Since $\tilde{\chi}_2(x) + \tilde{\chi}_3(x) + \tilde{\chi}_4(x) = -1$, by Proposition 1.4, (*) and (**), we get $\chi_4|_H = \tilde{\chi}_2 + \tilde{\chi}_3 + \tilde{\chi}_4$. Similarly, we may assume that $\chi_j|_H = \tilde{\chi}_{j-2} + \tilde{\chi}_{j-1} + \tilde{\chi}_j$ for $j=8, 12, 16$. We may also assume that $\chi_i|_H = \chi_{i+1}|_H = \chi_{i+2}|_H = \tilde{\chi}_i$ for $i=5, 9, 13$ using Frobenius reciprocity (*) and (**). This completes the proof of (i). Since $b_0 \cong B_0(S)$, by Lemma 1.16(i), $\tilde{\phi}_2(1) = \tilde{\phi}_3(1) = (q-1)/2$. Thus $I_G(\tilde{\phi}_j) = G$ for $j=1, 2, 3$ since $|G:H|=3$. For all $\phi_i \in \text{IBr}(B_0)$ we have $\phi_i|_H \in \text{IBr}(b_0)$ by Clifford's theorem since $|G:H|=3$. Thus, by [15, V 16.6 Satz], we get (ii) for suitable indexing of ϕ_2, \dots, ϕ_9 . This completes the proof of the lemma.

PROPOSITION 6.7. *Assume as in Lemma 6.6. Then the decomposition matrix D of B_0 is as follows.*

(i) $3 < q \equiv 3 \pmod{8}$:

$$\begin{array}{ccccccc}
 1 & 0 & 0 & & & & \\
 0 & 1 & 0 & & & & \\
 0 & 0 & 1 & & 0 & & 0 \\
 1 & 1 & 1 & & & & \\
 & & & 1 & 0 & 0 & \\
 & & & 0 & 1 & 0 & \\
 & 0 & & 0 & 0 & 1 & 0 \\
 D = & & & 1 & 1 & 1 & \\
 & & & & & & 1 & 0 & 0 \\
 & 0 & & 0 & & & 0 & 1 & 0 \\
 & & & & & & 0 & 0 & 1 \\
 & & & & & & 1 & 1 & 1 \\
 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1.
 \end{array}$$

(ii) $3 < q \equiv 5 \pmod{8}$:

$$\begin{array}{ccccccc}
 1 & 0 & 0 & & & & \\
 0 & 1 & 0 & & & & \\
 0 & 0 & 1 & & 0 & & 0 \\
 1 & 1 & 1 & & & & \\
 1 & 0 & 0 & 1 & 0 & 0 & \\
 0 & 1 & 0 & 0 & 1 & 0 & \\
 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
 D = & 1 & 1 & 1 & 1 & 1 & 1 \\
 1 & 0 & 0 & & & & 1 & 0 & 0 \\
 0 & 1 & 0 & & & & 0 & 1 & 0 \\
 0 & 0 & 1 & & 0 & & 0 & 0 & 1 \\
 1 & 1 & 1 & & & & 1 & 1 & 1 \\
 1 & 0 & 0 & & & & 1 & 0 & 0 \\
 0 & 1 & 0 & (*) & & & 0 & 1 & 0 \\
 0 & 0 & 1 & & & & 0 & 0 & 1 \\
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1,
 \end{array}$$

where (*) is one of the following types

$$\begin{array}{cccccccc}
 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 1, & 0 & 1 & 0, & 0 & 0 & 1, & 1 & 0 & 0, & 0 & 1 & 0, & 1 & 0 & 0.
 \end{array}$$

$$(1) \quad \tilde{D}B=LD,$$
$$\begin{array}{cccc} & & 1 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & 1 & 1 & 1 \end{array} \quad \begin{array}{c} \\ \\ \\ \end{array} \quad \begin{array}{c} \\ \\ \\ \end{array} \quad \begin{array}{c} \\ \\ \\ \end{array}$$

and

By Lemma 6.6,

$$(4) \quad D^{xz} = (d_{il}^{xz})_i = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

where D^{xz} is the generalized decomposition matrix of B_0 relative to xz . Clearly, S is normal in $C_G(x)$. By the proof of Lemma 3.1, $e(C_G(x))=3$. Since $q \equiv 3 \pmod{8}$, by Lemmas 2.3 and 6.5(2),

$$(5) \quad (d_{\nu_i, \alpha}^x)_{i, \alpha} = \begin{array}{cccc} & \phi_1^x & \phi_2^x & \phi_3^x \\ 1_G = \chi_{\nu_1} & 1 & 0 & 0 \\ \chi_{\nu_2} & \bar{\delta}_2 & 0 & 0 \\ \chi_{\nu_3} & \bar{\delta}_3 & 0 & 0 \\ \chi_{\nu_4} & \bar{\delta}_4 & 0 & 0 \\ \chi_{\nu_5} & 0 & \bar{\delta}_5 & 0 \\ \chi_{\nu_6} & 0 & \bar{\delta}_6 & 0 \\ \chi_{\nu_7} & 0 & \bar{\delta}_7 & 0 \\ \chi_{\nu_8} & 0 & \bar{\delta}_8 & 0 \\ \chi_{\nu_9} & 0 & 0 & \bar{\delta}_9 \\ \chi_{\nu_{10}} & 0 & 0 & \bar{\delta}_{10} \\ \chi_{\nu_{11}} & 0 & 0 & \bar{\delta}_{11} \\ \chi_{\nu_{12}} & 0 & 0 & \bar{\delta}_{12} \\ \chi_{\nu_{13}} & \bar{\delta}_{13} & \bar{\delta}_{13} & \bar{\delta}_{13} \\ \chi_{\nu_{14}} & \bar{\delta}_{14} & \bar{\delta}_{14} & \bar{\delta}_{14} \\ \chi_{\nu_{15}} & \bar{\delta}_{15} & \bar{\delta}_{15} & \bar{\delta}_{15} \\ \chi_{\nu_{16}} & \bar{\delta}_{16} & \bar{\delta}_{16} & \bar{\delta}_{16} \end{array}$$

where $d_{i\alpha}^x$ are the generalized decomposition numbers of B_0 relative to x , $\bar{\delta}_i = \pm 1$, $\chi_{\nu_1} = \chi_1 = 1_G$, $\{\chi_{\nu_2}, \dots, \chi_{\nu_{16}}\} = \{\chi_2, \dots, \chi_{16}\}$, $\{\phi_1^x = 1_M, \phi_2^x, \phi_3^x\} = \text{IBr}(b_x)$ and $M = C_G(x)$. Let $c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in $L_2(q)$. Then, by [10, Theorem 38.1] and Lemma 1.16, we may assume that

$$(6) \quad \phi_2^x(c) = (-1 + \sqrt{-q})/2 \quad \text{and} \quad \phi_3^x(c) = (-1 - \sqrt{-q})/2.$$

By Lemma 6.6 and (5), $\{\chi_{\nu_1}, \dots, \chi_{\nu_4}\} = \{\chi_1, \dots, \chi_4\}$ and $\{\chi_{\nu_{13}}, \dots, \chi_{\nu_{16}}\} = \{\chi_{13}, \dots, \chi_{16}\}$. We may assume that $\chi_5 \in \{\chi_{\nu_5}, \dots, \chi_{\nu_8}\}$. By Lemma 6.6, $\chi_5|_H = \chi_6|_H = \chi_7|_H$. Thus, by (5) and (6), we get that χ_6 and χ_7 are both in $\{\chi_{\nu_5}, \dots, \chi_{\nu_8}\}$. Similarly, none of $\{\chi_9, \chi_{10}, \chi_{11}\}$ are in $\{\chi_{\nu_5}, \dots, \chi_{\nu_8}\}$. Hence, by (2), (5) and (6), we know $\{\chi_{\nu_5}, \dots, \chi_{\nu_8}\} = \{\chi_5, \dots, \chi_8\}$. Thus, $\{\chi_{\nu_9}, \dots, \chi_{\nu_{12}}\} = \{\chi_9, \dots, \chi_{12}\}$. Hence we may assume that $\chi_{\nu_i} = \chi_i$ for all $i=1, \dots, 16$. Therefore, by Lemma 6.6,

$$\begin{array}{ccc} \phi_1^x & \phi_2^x & \phi_3^x \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{array}$$

$$(7) \quad (d_{i\alpha}^x)_{i,\alpha} = \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & -1. \end{array}$$

Next, we want to know the generalized decomposition numbers $d_{i\alpha}^z$ of B_0 relative to z . Let $L = C_G(z)$. As for x , $e(L) = 3$ and $l(B_0(L)) = 3$. Since $N_L(P) = \langle t, C_L(P) \rangle$ and $z^t = z$, we get from Proposition 6.1 and Theorem 2.4(1) that $k(B_0(L)) = k(b_z) = 16$. By Theorem 2.4(2) and Lemmas 6.6 and 6.5, L is solvable, so that $b_z \cong B_0(P \cdot Z_3)$ from Proposition 6.1 where $P \cdot Z_3$ is a semi-direct product of its normal subgroup P by Z_3 and it is not the direct product $P \times Z_3$. Thus, by Lemma 6.5,

$$(8) \quad (d_{\nu_i, \alpha}^z)_{i,\alpha} = \begin{array}{cccc} & \phi_1^z & \phi_2^z & \phi_3^z \\ 1_G = \chi_{\nu_1} & 1 & 0 & 0 \\ \chi_{\nu_2} & 0 & \delta_2 & 0 \\ \chi_{\nu_3} & 0 & 0 & \delta_3 \\ \chi_{\nu_4} & \delta_4 & \delta_4 & \delta_4 \\ \chi_{\nu_5} & \delta_5 & 0 & 0 \\ \chi_{\nu_6} & 0 & \delta_6 & 0 \\ \chi_{\nu_7} & 0 & 0 & \delta_7 \\ \chi_{\nu_8} & \delta_8 & \delta_8 & \delta_8 \\ \chi_{\nu_9} & \delta_9 & 0 & 0 \\ \chi_{\nu_{10}} & 0 & \delta_{10} & 0 \\ \chi_{\nu_{11}} & 0 & 0 & \delta_{11} \\ \chi_{\nu_{12}} & \delta_{12} & \delta_{12} & \delta_{12} \\ \chi_{\nu_{13}} & \delta_{13} & 0 & 0 \\ \chi_{\nu_{14}} & 0 & \delta_{14} & 0 \\ \chi_{\nu_{15}} & 0 & 0 & \delta_{15} \\ \chi_{\nu_{16}} & \delta_{16} & \delta_{16} & \delta_{16} \end{array}$$

where $\delta_i = \pm 1$, $\chi_{\nu_1} = \chi_1 = 1_G$, $\{\chi_{\nu_2}, \dots, \chi_{\nu_{16}}\} = \{\chi_2, \dots, \chi_{16}\}$ and $\{\phi_1^z, \phi_2^z, \phi_3^z\} = \text{IBr}(b_z)$. Clearly, $\phi_1^z(1) = \phi_2^z(1) = \phi_3^z(1) = 1$. Hence, by Lemma 6.6 and (8),

$$(9) \quad \begin{aligned} \{\chi_{\nu_4}, \chi_{\nu_8}, \chi_{\nu_{12}}, \chi_{\nu_{16}}\} &= \{\chi_4, \chi_8, \chi_{12}, \chi_{16}\} \\ \{\delta_4, \delta_8, \delta_{12}, \delta_{16}\} &= \{1, 1, 1, -1\}. \end{aligned}$$

By Lemma 6.6, $\chi_i(z) = \chi_{i+1}(z) = \chi_{i+2}(z) = 1$ for $i=1, 5, 9$. So it follows from (4), (7), (8) and [10, Theorem 63.3] that $\delta_4 = \delta_8 = \delta_{12} = 1$ and $\delta_{16} = -1$. Thus, again by (4), (7), (8) and [10, Theorem 63.3],

$$(10) \quad \begin{array}{ccc} \phi_1^z & \phi_2^z & \phi_3^z \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ (d_{i\alpha}^z)_{i,\alpha} = & 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & -1 & -1 \end{array}$$

for suitable indexing. By (2), (3), (10) and [10, Theorem 63.3],

$$D = \begin{array}{cccccccccc} \chi_{13} & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ \chi_{14} & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ \chi_{15} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \chi_{16} & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1. \end{array}$$

This completes the proof of (i).

(ii) Since $b_0 \cong B_0(S)$, as in the proof of (i) we get

$$\begin{array}{ccccccc}
 & & 1 & 0 & 0 & & \\
 & & 0 & 1 & 0 & & \\
 & & 0 & 0 & 1 & & \\
 & & 1 & 1 & 1 & & \\
 & & ? & & ? & & \\
 (11) \quad D = & 1 & 1 & 1 & 1 & 1 & 1 \\
 & & ? & & 0 & & ? \\
 & & 1 & 1 & 1 & & 1 & 1 & 1 \\
 & & ? & & ? & & ? \\
 & & 1 & 1 & 1 & 1 & 1 & 1 & 1
 \end{array}$$

and

$$\begin{aligned}
 (12) \quad & d_{5\lambda} + d_{6\lambda} + d_{7\lambda} = 1 \quad \text{for } \lambda = 1, \dots, 6 \\
 & d_{9\lambda} + d_{10,\lambda} + d_{11,\lambda} = 1 \quad \text{for } \lambda = 1, 2, 3, 7, 8, 9 \\
 & d_{13,\lambda} + d_{14,\lambda} + d_{15,\lambda} = 1 \quad \text{for } \lambda = 1, \dots, 9.
 \end{aligned}$$

As in the proof of (i), we have

$$\begin{array}{ccccccc}
 & & \phi_1^x & \phi_2^x & \phi_3^x & & \phi_1^z & \phi_2^z & \phi_3^z \\
 1 & & 1 & 0 & 0 & & 1 & 0 & 0 \\
 1 & & 1 & 0 & 0 & & 0 & 1 & 0 \\
 1 & & 1 & 0 & 0 & & 0 & 0 & 1 \\
 -1 & & -1 & 0 & 0 & & 1 & 1 & 1 \\
 -1 & & 1 & 1 & 0 & & -1 & 0 & 0 \\
 -1 & & 1 & 1 & 0 & & 0 & -1 & 0 \\
 -1 & & 1 & 1 & 0 & & 0 & 0 & -1 \\
 (13) \quad (d_{ii}^{xz})_i = & 1 & (d_{i\alpha}^x)_{i,\alpha} = -1 & -1 & 0 & (d_{i\alpha}^z)_{i,\alpha} = -1 & -1 & -1 \\
 -1 & & 1 & 0 & 1 & & -1 & 0 & 0 \\
 -1 & & 1 & 0 & 1 & & 0 & -1 & 0 \\
 -1 & & 1 & 0 & 1 & & 0 & 0 & -1 \\
 1 & & -1 & 0 & -1 & & -1 & -1 & -1 \\
 1 & & 1 & 1 & 1 & & 1 & 0 & 0 \\
 1 & & 1 & 1 & 1 & & 0 & 1 & 0 \\
 1 & & 1 & 1 & 1 & & 0 & 0 & 1 \\
 -1, & & -1 & -1 & -1, & & 1 & 1 & 1
 \end{array}$$

where d_{ii}^{xz} , $d_{i\alpha}^x$, $d_{i\alpha}^z$, ϕ_α^x and ϕ_α^z are the same as in (i). By Lemmas 6.6 and 1.16, we know the degrees of all χ_i and ϕ_α . Thus, by (11), (12), (13) and [10, Theorem 63.3], we may assume

Similarly, we may assume

So, by (11), (12), (13), (14), (15) and [10, Theorem 63.3],

Thus, considering the degrees of χ_i and ϕ_α , by (11)–(16) we get the following six cases:

Thus, for suitable indexing of χ_i and ϕ_α , we obtain (ii).

The following theorem is one of the main results of this section.

THEOREM 6.8. *Let $\bar{G}=G/O(G)$ and $S=O'(\bar{G})$. If G is nonsolvable, $e=9$ and $e(S) \neq 9$, then we have the following.*

- (i) $S \cong Z_2 \times Z_2 \times L_2(q)$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$.
(ii) For any subnormal subgroup \bar{L} of \bar{G} of odd index with $e(\bar{L})=9$, $B_0 \cong B_0(\bar{L})$.

PROOF. We may assume $O(G)=1$ by Lemma 1.1, so that $S=O'(G)$.

(i) Since G is nonsolvable, $e(S)=3$. Thus, we get (i) from Proposition 1.10 and Lemma 1.12.

(ii) Firstly, we want to prove that

$$(*) \quad \begin{cases} \text{if } L \text{ is a normal subgroup of } G \text{ such that } |G:L| \text{ is an odd prime and} \\ e(L)=9 \text{ and if } H \text{ is a normal subgroup of } L \text{ such that } |L:H| \text{ is an odd} \\ \text{prime and } e(H)=3, \text{ then } B_0 \cong B_0(L). \end{cases}$$

Let $b_0 = B_0(L)$. By Lemma 3.1, $k(B_0) = k(b_0) = 16$ and $l(B_0) = l(b_0) = 9$. We may write $O'(G) = S = \langle x, y \rangle \times L_2(q)$ and $P = \langle x, y, z, w \rangle$ where $\langle z, w \rangle$ is a Sylow 2-subgroup of $L_2(q)$. By the proof of Lemma 6.6, we may assume $x^s = x$, $y^s = y$, $z^s = w$, $w^s = zw$, $x^t = y$, $y^t = xy$, $z^t = z$ and $w^t = w$ where $s, t \in N_L(P) = \langle s, t, C_L(P) \rangle$. So $N_G(P) = \langle s, t, C_G(P) \rangle$. By the proof of Lemma 3.1, $l(b_{xz}) = 1$. Thus, by Lemma 1.9, $\chi_i(xz) = \pm 1$ for all $\chi_i \in \text{Irr}(B_0)$. By Lemma 6.6, we know the values $\tilde{\chi}_f(1)$ and $\tilde{\chi}_j(xz)$ for all $\tilde{\chi}_j \in \text{Irr}(b_0)$. Using this, if $\tilde{\chi}, \tilde{\chi}' \in \text{Irr}(b_0)$ and $\tilde{\chi}(1) = \tilde{\chi}'(1)$, then $\tilde{\chi}(xz) = \tilde{\chi}'(xz) = \pm 1$. Hence it follows from Corollary 1.8 that $B_0 \cong b_0$. Thus we get (*). On the other hand, G/S is solvable from [12, Theorem]. Hence we can verify (ii) by repeating the above way. This completes the proof.

PROPOSITION 6.9. *Let D be the decomposition matrix of B_0 , and let $S = O'(G/O(G))$. If $e=9$, then we have following.*

(i) When G is solvable,

$$D = \begin{pmatrix} 1 & 0 & 0 & & & & & & & \\ 0 & 1 & 0 & & 0 & & & & & \\ 0 & 0 & 1 & & & & & & & \\ & & & 1 & 0 & 0 & & & & \\ & 0 & & 0 & 1 & 0 & & & 0 & \\ & & & 0 & 0 & 1 & & & & \\ & & & & & & 1 & 0 & 0 & \\ & 0 & & & & & 0 & 1 & 0 & \\ & & & & & & 0 & 0 & 1 & \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \end{pmatrix}$$

(ii) When G is nonsolvable and $e(S)=9$, we know D from Theorem 3.4(2)

and Lemma 1.16(ii).

(iii) When G is nonsolvable and $e(S)=3$, we know D from Theorem 6.8 and Proposition 6.7.

REMARK 1. There is a finite group G with an elementary abelian Sylow 2-subgroup P of order 16 such that $e(G)=9$ and $O'(G/O(G)) \cong Z_2 \times Z_2 \times L_2(q)$ for $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$. Let $\langle z, w \rangle$ be a Sylow 2-subgroup of $L_2(q)$, and let $S = \langle x, y \rangle \times L_2(q)$ and $P = \langle x, y, z, w \rangle$ where $\langle x, y \rangle$ is elementary abelian of order 4. There is an automorphism r of $\langle x, y \rangle$ with $x^r = y$ and $y^r = xy$. We can consider that $r \in \text{Aut}(S)$ if its we consider that r is trivial on $L_2(q)$. So there is a semi-direct product G of its normal subgroup S by $\langle r \rangle$. Then, $e(G)=9$ and $O'(G) = S = Z_2 \times Z_2 \times L_2(q)$.

The next theorem is one of the main results of this section.

THEOREM 6.10. If G is nonsolvable and $e=15$, then $B_0 \cong B_0(SL(2, 16))$.

PROOF. By Lemma 1.1, we may assume $O(G)=1$. Let $S=O'(G)$. Since G is nonsolvable, it follows from Proposition 1.10 and Lemma 1.12 that $e(S) \neq 1$ and $e(S) \neq 5$. So that $e(S)=3$ or 15 . Firstly, suppose $e(S)=3$. By Proposition 1.10 and Lemma 1.12, $S \cong Z_2 \times Z_2 \times L_2(q)$ for some $q > 3$ with $q \equiv 3$ or $5 \pmod{8}$. Thus, there is an involution $x \in P \cap Z(S)$. We can write $N_S(P) = \langle s, C_S(P) \rangle$ for some $s \in N_S(P)$. Thus, $x^s = x$. Since $e=15$, we can write $N_G(P) = \langle t, C_G(P) \rangle$ for some $t \in N_G(P)$. Since $N_S(P)/C_S(P)$ can be considered as a subgroup of $N_G(P)/C_G(P)$ through the canonical monomorphism, we get that $s \equiv t^{5i} \pmod{C_G(P)}$ for some integer i with $i \not\equiv 0 \pmod{3}$. Thus, $x = x^{t^{5i}}$. This is a contradiction. Hence $e(S)=15$, so that $S \cong SL(2, 16)$ from Proposition 1.10 and Lemma 1.12.

We prove $B_0 \cong B_0(S)$ by induction on $|G|$. Let $G \neq S$. Since G/S is solvable by [12, Theorem], G has a normal subgroup H of odd prime index with $S \subseteq H$. Let $b_0 = B_0(H)$, and let z be an involution in P . Since $b_0 \cong B_0(S)$ by induction, we get $k(b_0)=16$ and

$$(*) \quad \tilde{\chi}_i(1) = \begin{cases} 1 & \text{for } i=1 \\ 15 & \text{for } i=2, \dots, 9 \\ 17 & \text{for } i=10, \dots, 16, \end{cases} \quad \tilde{\chi}_i(z) = \begin{cases} 1 & \text{for } i=1 \\ -1 & \text{for } i=2, \dots, 9 \\ 1 & \text{for } i=10, \dots, 16 \end{cases}$$

using [10, Theorem 38.2], where $\{\tilde{\chi}_1, \dots, \tilde{\chi}_{16}\} = \text{Irr}(b_0)$. Since all involutions in P are G -conjugate, $P \cap G' = P$ by [10, Theorem 18.4]. Thus, $k'(B_0)$ is odd from Proposition 1.5. Now, we want to claim that $k(B_0)=16$. If $k'(B_0)=1$, we get from Propositions 1.5 and 1.6 that $k(B_0)=16$. Suppose $k(B_0) \neq 16$. Since $e=15$,

$l(b_z)=1$. Thus, by Lemma 1.15(2), $k(B_0)=8$. So that $k'(B_0)=3, 5$ or 7 . Let $\{\chi_1, \dots, \chi_8\}=\text{Irr}(B_0)$.

Case 1. $k'(B_0)=7$: We may assume $\chi_1(1)=\dots=\chi_7(1)=1$ and $\chi_8(1)>1$. By Clifford's theorem, Proposition 1.3 and (*), we have $\chi_1|_H=\dots=\chi_7|_H=\tilde{\chi}_1$. Thus, by Proposition 1.4, $(\chi_8|_H, \tilde{\chi}_j) \neq 0$ for $j=2, \dots, 16$. Then we have a contradiction from Clifford's theorem and (*) by considering the degrees of $\tilde{\chi}_j$.

Case 2. $k'(B_0)=5$: We may assume $\chi_i(1)=1$ for $i=1, \dots, 5$ and $\chi_j(1)>1$ for $j=6, 7, 8$. As in Case 1 we know $\chi_1|_H=\dots=\chi_5|_H=\tilde{\chi}_1$. Since $k(B_0) \neq k(b_0)$, $B_0 \not\cong b_0$. So that we get from Proposition 1.6 that $G \neq VH$ where V is a subgroup of G with $C_G(P)=P \times V$. Since $k'(B_0)=5$, $|G:H|=5$ by Proposition 1.5. So, by Clifford's theorem and Proposition 1.4,

$$\chi_6|_H = \tilde{\chi}_2 + \dots + \tilde{\chi}_6, \quad \chi_7|_H = \tilde{\chi}_7 + \dots + \tilde{\chi}_{11},$$

$$\chi_8|_H = \tilde{\chi}_{12} + \dots + \tilde{\chi}_{16}$$

for suitable indexing of $\tilde{\chi}_2, \dots, \tilde{\chi}_{16}$. Hence we have a contradiction from Clifford's theorem and (*) by considering the degrees of $\tilde{\chi}_j$.

Case 3. $k'(B_0)=3$: Let $\chi_i(1)=1$ for $i=1, 2, 3$ and $\chi_j(1)>1$ for $j=4, \dots, 8$. As in Case 2, $|G:H|=3$. Then, by Proposition 1.4, for suitable indexing of $\tilde{\chi}_2, \dots, \tilde{\chi}_{16}$, we get

$$\chi_4|_H = \tilde{\chi}_2 + \tilde{\chi}_3 + \tilde{\chi}_4, \quad \chi_5|_H = \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_7, \quad \chi_6|_H = \tilde{\chi}_8 + \tilde{\chi}_9 + \tilde{\chi}_{10}$$

$$\chi_7|_H = \tilde{\chi}_{11} + \tilde{\chi}_{12} + \tilde{\chi}_{13}, \quad \chi_8|_H = \tilde{\chi}_{14} + \tilde{\chi}_{15} + \tilde{\chi}_{16}$$

Then we have a contradiction as in Case 2.

Thus, $k(B_0)=16$. Let $\{\chi_1, \dots, \chi_{16}\}=\text{Irr}(B_0)$. Since $l(b_z)=1$, $\chi_i(z)=\pm 1$ for $i=1, \dots, 16$ from Lemma 1.9. Thus, we know from Clifford's theorem, Proposition 1.3 and (*) that $\chi_i|_H \in \text{Irr}(b_0)$ for all $i=1, \dots, 16$. Hence, by Proposition 1.4, we may assume that $\chi_i|_H = \tilde{\chi}_i$ for all $i=1, \dots, 16$. This shows $k'(B_0)=1$. So that $B_0 \cong b_0$ from Propositions 1.5 and 1.6. This completes the proof of the theorem.

PROPOSITION 6.11. *If $e=15$, then there is a basic set W of B_0 such that W contains the trivial Brauer character and the decomposition matrix of B_0 with respect to W has the form*

$$\begin{array}{ccccccc} 1_G & 1 & & & & & \\ & \delta_2 & & & & & 0 \\ & & \cdot & & & & \\ & & & \cdot & & & \\ & 0 & & & \cdot & & \\ & & & & & \delta_{15} & \\ \delta_{16} & \delta_{16} & \cdot & \cdot & \cdot & \delta_{16} & \end{array}$$

where $\delta_i = \pm 1$.

PROOF. The proof is similar to that of Proposition 5.2 (cf. the case when $e=7$ in Proposition 5.2).

LEMMA 6.12. *If $e=21$, then there is an involution $z \in P$ and there are two elements $s, t \in N_G(P)$ such that $N_G(P) = \langle s, t, C_G(P) \rangle$, $z^s = z$ and $z^t = z$.*

PROOF. Firstly, we want to prove that

$$(*) \quad \begin{cases} \text{there is an involution } u \in P \text{ and there are two elements } s, t \in N_G(P) \text{ such} \\ \text{that } N_G(P) = \langle s, t, C_G(P) \rangle, s \text{ and } t \text{ have orders 3 and 7 modulo } C_G(P) \\ \text{respectively, and } u^s = u. \end{cases}$$

We may assume $O(G)=1$ by the proof of Lemma 1.1. Since $S=O'(G)$ is normal in G , $e(S)=1, 7$ or 21 . When $e(S)=7$ or 21 , we get (*) from Proposition 1.10 and Lemma 1.12. Assume $e(S)=1$. Then P is normal in G and $|G:P|=21$. We can write $G = \langle s, t, P \rangle$ for $s, t \in G$ such that s and t have orders 3 and 7 modulo P , respectively. Clearly, there is an involution $y \in P$ with $y^t = y$. Suppose $x^s \neq x$ for all involutions $x \in P$. Then, $e(C_G(y))=7$. By Proposition 5.2 and [10, Lemma 66.1], the Cartan matrix of b_y has 2 as an elementary divisor of multiplicity 6. Thus, by [5, (7G)], [18, Proposition 1.2] and [10, Theorem 65.4], we get $l(B_0) \geq 7$ since all involutions in G are conjugate. On the other hand, $l(B_0)=5$ since G/P is noncyclic of order 21 (cf. Lemma 1.13). This is a contradiction. Hence we obtain (*).

Next, we prove the lemma. There is an involution $z \in P$ with $z^t = z$. By (*), there are other two involutions $v, w \in P$ such that $v^s = v$, $w^s = w$ and u, v, w are all distinct. It suffices to show $z \in \{u, v, w\}$. Suppose $z \notin \{u, v, w\}$. Since $z^s \neq z$, we know that $\{1, u, v, w, z, uz, vz, wz\}$ is the set of all representatives of $\langle s \rangle$ -conjugate classes of P . Since $u \neq z$, we get $u^t \neq u$. Thus, $\{1, u, uz, z\}$ is the set of all representatives of $\langle t \rangle$ -conjugate classes of P . Hence, by elementary calculation, we get that $v \in \{uz, u^t z, \dots, u^{t^6} z\}$. Hence no two elements in $\{u, v, z\}$ are conjugate in G . These show that all G -conjugate classes of P are $\{1\}$, $\{z\}$, $\{u, u^t, \dots, u^{t^6}\}$ and $\{uz, u^t z, \dots, u^{t^6} z\}$. Thus, $z^s = z$. This is a contradiction. This completes the proof.

LEMMA 6.13. *If $e=21$, then $k(B_0)=16$ and $l(B_0)=5$.*

PROOF. Let s, t and z be the same as in Lemma 6.12. Hence s and t have orders 3 and 7 modulo $C_G(P)$, respectively. There is an involution $x \in P$ with $x^s = x$ and $x^t \neq x$. Thus, $\{1, x, xz, z\}$ is the set of all representatives of conjugate

classes of G of 2-elements by [10, Lemma 18.5]. We may assume $O(G)=1$ by Lemma 1.1. By Z^* -theorem [10, Theorem 67.1], $z \in Z(G)$. These imply from [10, Theorems 68.4 and 65.4] that $k(B_0)=2l(B_0)+l(b_x)+l(b_{xz})$. Since $e(C_G(x))=3$, $l(b_x)=3$ by Theorem 2.4(1). Similarly, $l(b_{xz})=3$. Since $z \in Z(G)$, as in the proof of Lemma 2.2 we get from Lemma 4.1 and Proposition 1.2 that $l(B_0)=5$, so that $k(B_0)=16$.

LEMMA 6.14. Assume $e=21$, $O(G)=1$, $O'(G)=Z_2 \times SL(2, 8)$ and G has a normal subgroup H of odd prime index with $e(H)=7$. Then for any involution z in $SL(2, 8)$, we have

$$\begin{aligned}\chi_1(1) &= \cdots = \chi_6(1) = 1, & \chi_7(1) &= \cdots = \chi_{12}(1) = 7, \\ \chi_{13}(1) &= \chi_{14}(1) = 21, & \chi_{15}(1) &= \chi_{16}(1) = 27, \\ \chi_1(z) &= \cdots = \chi_6(z) = 1, & \chi_7(z) &= \cdots = \chi_{12}(z) = -1, \\ \chi_{13}(z) &= \chi_{14}(z) = -3, & \chi_{15}(z) &= \chi_{16}(z) = 3\end{aligned}$$

where $\{\chi_1=1_G, \chi_2, \dots, \chi_{16}\} = \text{Irr}(B_0)$.

PROOF. Let $b_0=B_0(H)$, $S=O'(G)=\langle w \rangle \times SL(2, 8)$ and $P=\langle w, x, y, z \rangle$ where $\langle x, y, z \rangle$ is a Sylow 2-subgroup of $SL(2, 8)$. As in the proof of Lemma 5.3, $G/H = \langle rH \rangle$ for some $r \in N_G(P)$, $C_G(P)=C_H(P)$ and $|G:H|=3$. We can write $N_S(P) = \langle t, C_S(P) \rangle$ for some $t \in N_S(P)$. Since $P \cap Z(S) = \langle w \rangle$, it follows from Lemma 6.12 and Z^* -theorem [10, Theorem 67.1] that $w \in Z(G)$. Then, by the proof of Lemma 6.13, we may assume that $z^t \neq z$, $z^r = z$ and $l(b_z)=3$. By the proof of Lemma 2.2, $l(B_0(C_S(z)))=1$. So that $C_S(z)$ is 2-nilpotent from [10, Corollary 65.3]. Hence $C_G(z)$ is solvable. Since $e(C_G(z))=3$, by Proposition 6.1, $b_z \cong B_0(P \cdot Z_3)$ where $P \cdot Z_3$ is a semi-direct product of its normal subgroup P by Z_3 and it is not the direct product $P \times Z_3$. Then, by Lemma 6.5, we know the generalized decomposition numbers of B_0 relative to z . This implies

$$(*) \quad \chi_i(z) = \begin{cases} \pm 1 & \text{for } i=1, \dots, 12 \\ \pm 3 & \text{for } i=13, \dots, 16 \end{cases}$$

for suitable indexing of χ_2, \dots, χ_{16} . By Lemma 2.3, $b_0 \cong B_0(S)$. So, by [10, Theorem 38.2],

$$\begin{aligned}(**) \quad \tilde{\chi}_1(1) &= \tilde{\chi}_2(1) = 1, & \tilde{\chi}_3(1) &= \cdots = \tilde{\chi}_{10}(1) = 7, \\ \tilde{\chi}_{11}(1) &= \cdots = \tilde{\chi}_{16}(1) = 9, \\ \tilde{\chi}_1(z) &= \tilde{\chi}_2(z) = 1, & \tilde{\chi}_3(z) &= \cdots = \tilde{\chi}_{10}(z) = -1, \\ \tilde{\chi}_{11}(z) &= \cdots = \tilde{\chi}_{16}(z) = 1\end{aligned}$$

where $\{\tilde{\chi}_1=1_H, \tilde{\chi}_2, \dots, \tilde{\chi}_{16}\}=\text{Irr}(b_0)$. We can write $C_G(P)=P \times V$. By Theorem 2.4 and Lemma 6.13, we get $l(B_0) \neq l(b_0)=l(B_0(S))$. Hence $B_0 \cong b_0$, so that $VH \neq G$ by Proposition 1.6. Thus, $|G:VH|=|G:H|=3$. Hence, by Proposition 1.5, $k'(B_0)$ is divisible by 3. Since $|G:H|=3$, it follows from Frobenius reciprocity, Proposition 1.3 and (**) that $k'(B_0) \leq 6$. By observing the conjugate classes of G of 2-elements, we know $P \cap G' \neq P$ from [10, Theorem 18.4]. Hence $|G:VG'|$ is divisible by 2. Thus, $k'(B_0)=|G:VG'|=6$ from Proposition 1.5. Then, by (*) and (**), we may assume that

$$\chi_1|_H=\chi_2|_H=\chi_3|_H=\tilde{\chi}_1, \quad \chi_4|_H=\chi_5|_H=\chi_6|_H=\tilde{\chi}_2.$$

Similarly, we may assume that

$$\begin{aligned} \chi_{13}|_H &= \tilde{\chi}_5 + \tilde{\chi}_6 + \tilde{\chi}_7, & \chi_{14}|_H &= \tilde{\chi}_8 + \tilde{\chi}_9 + \tilde{\chi}_{10}, \\ \chi_{15}|_H &= \tilde{\chi}_{11} + \tilde{\chi}_{12} + \tilde{\chi}_{13}, & \chi_{16}|_H &= \tilde{\chi}_{14} + \tilde{\chi}_{15} + \tilde{\chi}_{16}. \end{aligned}$$

Hence we may assume that

$$\chi_7|_H=\chi_8|_H=\chi_9|_H=\tilde{\chi}_3, \quad \chi_{10}|_H=\chi_{11}|_H=\chi_{12}|_H=\tilde{\chi}_4.$$

Therefore the lemma is proved by (*).

Now, we state the next theorem which is one of the main results of this section.

THEOREM 6.15. *Let $\bar{G}=G/O(G)$ and $S=O'(\bar{G})$. If G is nonsolvable, $e=21$ and $e(S) \neq 21$, then we have the following.*

- (i) $S \cong Z_2 \times SL(2, 8)$.
- (ii) *For any subnormal subgroup \bar{L} of \bar{G} of odd index with $e(\bar{L})=21$, $B_0 \cong B_0(\bar{L})$.*

PROOF. We can assume $O(G)=1$ by Lemma 1.1. Hence $S=O'(G)$.

- (i) By Lemma 1.13, $e(S)=7$. Hence, by Proposition 1.10 and Lemma 1.12, $S \cong Z_2 \times SL(2, 8)$.

- (ii) Firstly, we want to show that

$$(*) \quad \begin{cases} \text{if } L \text{ is a normal subgroup of } G \text{ such that } |G:L| \text{ is an odd prime and} \\ e(L)=21, \text{ and if } H \text{ is a normal subgroup of } L \text{ such that } |L:H| \text{ is an} \\ \text{odd prime and } e(H)=7, \text{ then } B_0 \cong B_0(L). \end{cases}$$

Let $b_0=B_0(L)$. By Lemma 6.13, $k(B_0)=k(b_0)=16$ and $l(B_0)=l(b_0)=5$. Let $S=O'(G)=\langle w \rangle \times SL(2, 8)$ and $P=\langle w, x, y, z \rangle$ where $\langle x, y, z \rangle$ is a Sylow 2-subgroup of $SL(2, 8)$. As in the proof of Lemma 6.14,

$$(**) \quad \chi_i(z) = \begin{cases} \pm 1 & \text{for } i=1, \dots, 12 \\ \pm 3 & \text{for } i=13, \dots, 16 \end{cases}$$

where $\{\chi_1, \dots, \chi_{16}\} = \text{Irr}(B_0)$. Let $\{\tilde{\chi}_1, \dots, \tilde{\chi}_{16}\} = \text{Irr}(b_0)$. By Lemma 6.14, we may assume that

$$\begin{aligned}
 & \tilde{\chi}_1(1) = \dots = \tilde{\chi}_6(1) = 1, \quad \tilde{\chi}_7(1) = \dots = \tilde{\chi}_{12}(1) = 7, \\
 & \tilde{\chi}_{13}(1) = \tilde{\chi}_{14}(1) = 21, \quad \tilde{\chi}_{15}(1) = \tilde{\chi}_{16}(1) = 27, \\
 (***) \quad & \tilde{\chi}_1(z) = \dots = \tilde{\chi}_6(z) = 1, \quad \tilde{\chi}_7(z) = \dots = \tilde{\chi}_{12}(z) = -1, \\
 & \tilde{\chi}_{13}(z) = \tilde{\chi}_{14}(z) = -3, \quad \tilde{\chi}_{15}(z) = \tilde{\chi}_{16}(z) = 3.
 \end{aligned}$$

Thus, as in the proof of Theorem 5.4, by (**), (***), Clifford's theorem and Proposition 1.4, we may assume that $\chi_i|_L = \tilde{\chi}_i$ for all $i=1, \dots, 16$. Hence we get $B_0 \cong b_0$ by Corollary 1.7. This proves (*). Since G/S is solvable by [12, Theorem], we can verify (ii).

REMARK 2. There is a finite group G with elementary abelian Sylow 2-subgroups of order 16 such that $e(G)=21$ and $O'(G/O(G)) \cong Z_2 \times SL(2, 8)$. We know it as in Remark 1 of §2.

PROPOSITION 6.16. *If $e=21$, then there is a basic set W of B_0 such that W contains the trivial Brauer character and the decomposition matrix of B_0 with respect to W has the form*

$$\begin{array}{rcccccc}
 1_G & 1 & 0 & 0 & 0 & 0 \\
 & \delta_2 & 0 & 0 & 0 & 0 \\
 & 0 & \delta_3 & 0 & 0 & 0 \\
 & 0 & \delta_4 & 0 & 0 & 0 \\
 & 0 & 0 & \delta_5 & 0 & 0 \\
 & 0 & 0 & \delta_6 & 0 & 0 \\
 & 0 & 0 & 0 & \delta_7 & 0 \\
 & 0 & 0 & 0 & \delta_8 & 0 \\
 & 0 & 0 & 0 & 0 & \delta_9 \\
 & 0 & 0 & 0 & 0 & \delta_{10} \\
 & \delta_{11} & 0 & 0 & \delta_{11} & \delta_{11} \\
 & \delta_{12} & 0 & 0 & \delta_{12} & \delta_{12} \\
 & 0 & \delta_{13} & 0 & \delta_{13} & \delta_{13} \\
 & 0 & \delta_{14} & 0 & \delta_{14} & \delta_{14} \\
 & 0 & 0 & \delta_{15} & \delta_{15} & \delta_{15} \\
 & 0 & 0 & \delta_{16} & \delta_{16} & \delta_{16}
 \end{array}$$

where $\delta_i = \pm 1$.

PROOF. We can verify the proposition as in Proposition 5.2.

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Department of Mathematics
Chiba University, Chiba, 260